# Support Material: Proof the Theorem 1

For Working Paper: AN ALTERNATIVE APPROACH TO COMPARE THREE STATIONARY CONCEPTS: A DUMMY JUDGMENT

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## Symbols and Dummy Adding

The symbols in here are following the literature, Selten and Chmura 2008 in AER. Figure 1 provides the payoff matrix and the symbols.

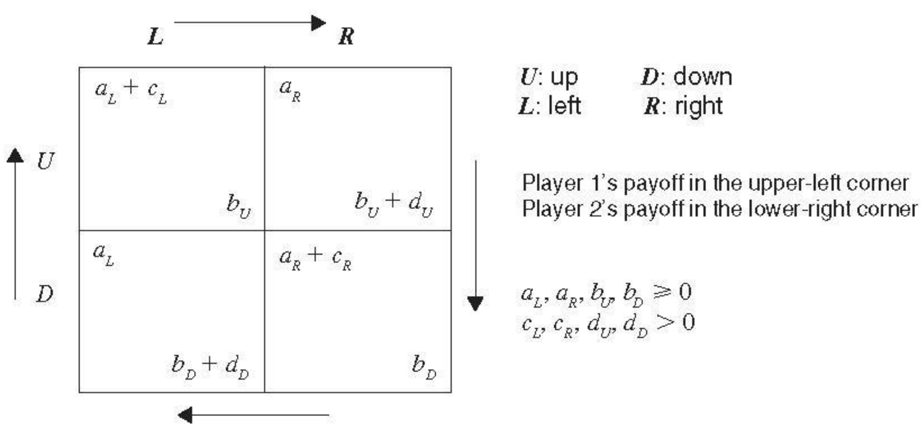


FIGURE 1. STRUCTURE OF THE EXPERIMENTAL 2x2-GAMES

**Figure 1** Payoff Matrix and the symbols (source: Selten, Reinhard and Thorsten Chmura. 2008. "Stationary Concepts for Experimental 2x2-Games." American Economic Review, 98(3), 938-66.). This figure here is to make this appendix to be more readable.

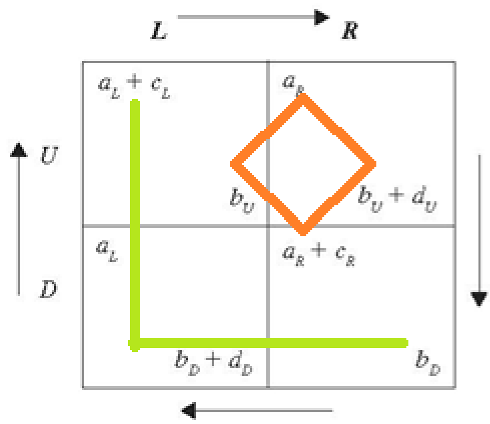


**Figure 2** The L-Shaped Dummy adding is add a same constant in the 4 matrix elements on the L-shape line (in green) in same time; similarly, the Diamond-Shaped Dummy adding is add a same constant in the 4 matrix elements on the Diamond-shape line (in red). This figure here is to make this appendix to be more readable.

## Theorem 1

Theorem 1: When a diamond Dummy is added on the payoff matrix of a game with complete mixed equilibrium ($\boldsymbol{c_L}, c_R, d_U, d_D > 0$ and $a_L, a_R, b_U, b_D \geq 0$ **),** we have $c^*_{L1} \leq c^*_{L2}$, $c^*_{R1} \geq c^*_{R2}$, $d^*_{U1} \geq d^*_{U2}$ and$d^*_{D1} \leq d^*_{D2}$, where $c^*_{L1}$ and $c^*_{L2}$ is $c^*_L$ in the original game and modified game (after adding the diamond dummy). When a L dummy added, the reverse happens.[1]

Before the proving the theorem 1, we need to prove two lemma first.

## Lemma 1

**LEMMA 1**: For $\forall\, A, B, a, b \in R$ and $A \geq a, B \geq b$,

(1) **We have and only have following sorts of the four number in terms of size:** $S_1 = A, a, B, b$; $S_2 = A, B, a, b$; $S_3 = X$; $S_4 = B, A, b, a$; $S_5 = B, b, A, a$, where $X = A, B, b, a$ if $A - a \geq B - b$; and, $X = B, A, a, b$ if $A - a \leq B - b$.

*(2)* **When we obtain** $B'$and $b'$ by adding a positive number $r$ on $B$ and $b$, that is, $B' = B + r$, $b' = b + r, r > 0$ and $r \in R$, if we define the sort of $A, B', a, b'$be $S_j$ and the original sort of $A, B, a, b$ be $S_i$, we have $j \geq i$.

[1] $\boldsymbol{c^*_L}$ can be obtained after impulse balance transformation(Selten, Reinhard and Thorsten Chmura. 2008. "Stationary Concepts for Experimental 2x2-Games." American Economic Review, 98(3), 938-66.)

Proof of lemma 1:

Because this lemma concerns all the possible sorts of only four numbers and we have another two conditions: $A > a$, and $B > b$, we can prove it by brute force method:

(1) For $A > a$, we can fix the relative position of two points $A$ and $a$ in a one-dimension real space. When $B > b$ is considered, we can list all the possible sorts in the following table(Table 1):

**Table 1. All possible Sorted Sequences and the 2nd Min. Value**

| $i$ of $S_i$ | | | ***A*** | | | ***a*** | | | | Max[min(A,b), min(a, B)] |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | ***A*** | | | ***a*** | *B* | *b* | *AaBb* | *B* |
| 2 | | | ***A*** | | *B* | ***a*** | | *b* | *ABab* | *a* |
| 3' | | | ***A*** | *B* | *b* | ***a*** | | | *ABba* | *b* |
| 3'' | | *B* | ***A*** | | | ***a*** | | *b* | *BAab* | *a* |
| 4 | | *B* | ***A*** | | *b* | ***a*** | | | *BAba* | *b* |
| 5 | *B* | *b* | ***A*** | | | ***a*** | | | *BbAa* | *A* |

To $S_3 = X$, if $B - b > A - a$, then, $S_3 \neq S_{3'}$, for we have $a > b$. So, we obtain $S_3 = S_{3''}$. Similarly, when $B - b < A - a$, $S_3 \neq S_{3''}$, for we have $a < b$. Therefore, $S_3 = S_{3'}$.

***(2)*** According to table 1, in which we list all the possible sorts of the four numbers, when $A$ and $a$ are fixed, let $S_j$ be the sort of $A, B', a, b'$ and $S_i$ be the original sort of $A, B, a, b$, we have $j \geq i$ if $B' > B$ or $b' > b$.

# Lemma 2

**LEMMA 2:** According to the definition of Equation12 by Selten, R. and T. Chmura(2008), for $\forall\ A, B, a, b \in R$, we have $S_l = \max[\min(A, a), \min(b, B)] = min2(A, B, a, b)$; $min2$ is the 2$^{nd}$ lowest value of the four matrix elements of the role (Up-Down or Right-Left)

# Proof of theorem 1:

Here, we choose Player 1 for example. By lemma 2, when the sort of the four number is$S_1$, we have the aspiration value (level) of Player 1 will be AV=B=A22.

For convenience, we use the following payoff matrix:

$$\textbf{payoff matrix} = \begin{vmatrix} A_{11} & & A_{12} & \\ & B_{12} & & B_{22} \\ A_{21} & & A_{22} & \\ & B_{11} & & B_{21} \end{vmatrix}$$

We also can list all possible aspiration value of player 1 in Table 2.

[Insert Table 2 Here]

For any game with completely mixed equilibrium, we have $A > a$ $(A_{11} > A_{21})$. According to Table 2, form $S_1$ to $S_2$, $c_L{}^*$keeps constant; form $S_2$ to $S_3$, $c_L{}^*$increases by $1/2\,(A_{12} - A_{21})$, which is positive in $S_2$; form $S_3$ to $S_4$, $c_L{}^*$ increases to $A_{11} - A_{21}$ form $1/2\,(A_{11} - A_{21}) + 1/2\,(A_{12} - A_{21})$. Thus, we obtain that $c_L{}^*$ is Monotone Increasing on $S_i$. Accordingly, $d_D^*$ is Monotone Increasing, $c_R{}^*$ and $d_U^*$ Monotone Decreasing on $S_i$.

For any game with completely mixed equilibrium, we have $A > a$ $(A_{11} > A_{21})$. If a diamond Dummy is added on the payoff matrix, to player 1, we have $r(\,r > 0)$ be added on B and b (B>b). By lemma 1, the sort of the four numbers in the Player 1's payoff matrix, $S_i$ might be modified by adding the diamond Dummy. Thus, we obtain Theorem 1.

## Table 2 Translated Matrix Element of the Sorted Sequences

| | | | *A* | *B* | *C* | *D* |
|---|---|---|---|---|---|---|
| | | *Sequences Type* | $S_1$ | $S_2$ | $S_3$ | $S_4$ |
| | | *2nd Min. Value* | $A_{22}$ | $A_{21}$ | $A_{12}$ | $A_{11}$ |
| | 1 | ${A_{11}}^*$ | $1/2\,(A_{11}-A_{22})+A_{22}$ | $1/2\,(A_{12}-A_{21})+A_{21}$ | $1/2\,(A_{11}-A_{12})+A_{12}$ | $A_{11}$ |
| | 2 | ${A_{21}}^*$ | $1/2\,(A_{21}-A_{22})+A_{22}$ | $A_{21}$ | $A_{21}$ | $A_{21}$ |
| | 3 | ${A_{22}}^*$ | $A_{22}$ | $1/2\,(A_{22}-A_{21})+A_{21}$ | $1/2\,(A_{22}-A_{12})+A_{12}$ | $1/2\,(A_{22}-A_{11})+A_{11}$ |
| | 4 | ${A_{12}}^*$ | $A_{12}$ | $A_{12}$ | $A_{12}$ | $1/2\,(A_{12}-A_{11})+A_{11}$ |
| | 5 | ${c_L}^* = {A_{11}}^* - {A_{21}}^*$ | $1/2\,(A_{11}-A_{21})$ | $1/2\,(A_{11}-A_{21})$ | $1/2\,(A_{11}-A_{21})$<br>$+\,1/2\,(A_{12}-A_{21})$ | $A_{11}-A_{21}$ |
| | 6 | ${c_R}^* = {A_{22}}^* - {A_{12}}^*$ | $A_{22}-A_{12}$ | $1/2\,(A_{22}-A_{12})$<br>$+\,1/2\,(A_{21}-A_{12})$ | $1/2\,(A_{22}-A_{12})$ | $1/2\,(A_{22}-A_{12})$ |
| | 7 | $d_D^* = {B_{11}}^* - {B_{21}}^*$ | $1/2\,(B_{11}-B_{21})$ | $1/2\,(B_{11}-B_{21})$ | $1/2\,(B_{11}-B_{21})$<br>$+\,1/2\,(B_{12}-B_{21})$ | $B_{11}-B_{21}$ |
| | 8 | $d_U^* = {B_{22}}^* - {B_{12}}^*$ | $B_{22}-B_{12}$ | $1/2\,(B_{22}-B_{12})$<br>$+\,1/2\,(B_{21}-B_{12})$ | $1/2\,(B_{22}-B_{12})$ | $1/2\,(B_{22}-B_{12})$ |