

Support Material: Proof the Theorem 1

For Working Paper: AN ALTERNATIVE APPROACH TO COMPARE THREE STATIONARY CONCEPTS: A DUMMY JUDGMENT

By Qiqi Cheng, Bin Xu and Zhijian Wang
Experimental social science laboratory, Zhejiang University

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Symbols and Dummy Adding

The symbols in here are following the literature, Selten and Chmura 2008 in AER. Figure 1 provides the payoff matrix and the symbols.

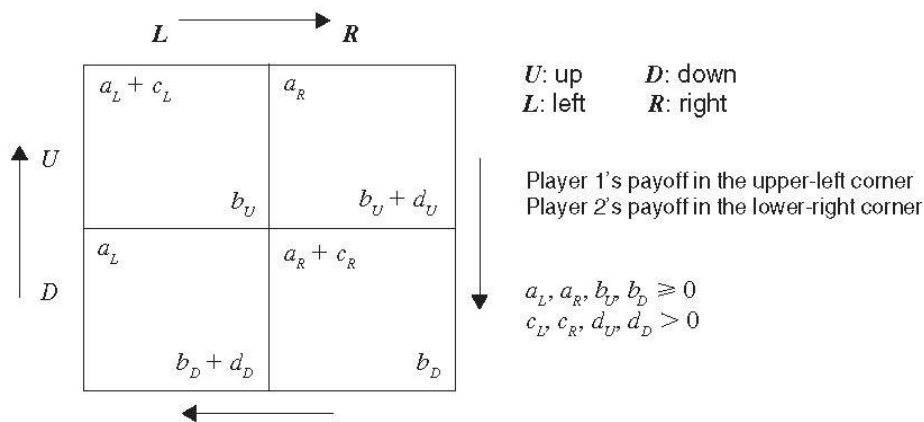


FIGURE 1. STRUCTURE OF THE EXPERIMENTAL 2X2-GAMES

Figure 1 Payoff Matrix and the symbols (source: Selten, Reinhard and Thorsten Chmura. 2008. "Stationary Concepts for Experimental 2x2-Games." American Economic Review, 98(3), 938-66.). This figure here is to make this appendix to be more readable.

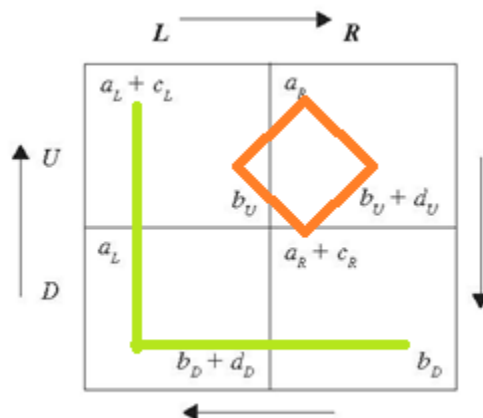


Figure 2 The L-Shaped Dummy adding is add a same constant in the 4 matrix elements on the L-shape line (in green) in same time; similarly, the Diamond-Shaped Dummy adding is add a same constant in the 4 matrix elements on the Diamond-shape line (in red). This figure here is to make this appendix to be more readable.

Theorem 1

Theorem 1: When a diamond Dummy is added on the payoff matrix of a game with complete mixed equilibrium ($c_L, c_R, d_U, d_D > 0$ and $a_L, a_R, b_U, b_D \geq 0$), we have $c_{L1}^* \leq c_{L2}^*$, $c_{R1}^* \geq c_{R2}^*$, $d_{U1}^* \geq d_{U2}^*$ and $d_{D1}^* \leq d_{D2}^*$, where c_{L1}^* and c_{L2}^* is c_L^* in the original game and modified game (after adding the diamond dummy). When a L dummy added, the reverse happens.¹

Before the proving the theorem 1, we need to prove two lemma first.

Lemma 1

LEMMA 1: For $\forall A, B, a, b \in R$ and $A \geq a, B \geq b$,

- (1) **We have and only have following sorts of the four number in terms of size:** $S_1 = A, a, B, b$; $S_2 = A, B, a, b$; $S_3 = X$; $S_4 = B, A, b, a$; $S_5 = B, b, A, a$, where $X = A, B, b, a$ if $A - a \geq B - b$; and, $X = B, A, a, b$ if $A - a \leq B - b$.
- (2) **When we obtain B' and b' by adding a positive number r on B and b , that is, $B' = B + r$, $b' = b + r$, $r > 0$ and $r \in R$, if we define the sort of A, B', a, b' be S_j and the original sort of A, B, a, b be S_i , we have $j \geq i$.**

¹ c_L^* can be obtained after impulse balance transformation (Selten, Reinhard and Thorsten Chmura. 2008. "Stationary Concepts for Experimental 2x2-Games." American Economic Review, 98(3), 938-66.)

Proof of lemma 1:

Because this lemma concerns all the possible sorts of only four numbers and we have another two conditions: $A > a$, and $B > b$, we can prove it by brute force method:

- (1) For $A > a$, we can fix the relative position of two points A and a in a one-dimension real space. When $B > b$ is considered, we can list all the possible sorts in the following table (Table 1):

Table 1. All possible Sorted Sequences and the 2nd Min. Value

i of S_i			A			a				$\text{Max}[\min(A,b), \min(a, B)]$
1			A			a	B	b	$AaBb$	B
2			A		B	a		b	$ABab$	a
3'			A	B	b	a			$ABba$	b
3''		B	A			a		b	$BAab$	a
4		B	A		b	a			$BAba$	b
5	B	b	A			a			$BbAa$	A

To $S_3 = X$, if $B - b > A - a$, then, $S_3 \neq S_{3'}$, for we have $a > b$. So, we obtain $S_3 = S_{3''}$. Similarly, when $B - b < A - a$, $S_3 \neq S_{3''}$, for we have $a < b$. Therefore, $S_3 = S_{3'}$.

- (2) According to table 1, in which we list all the possible sorts of the four numbers, when A and a are fixed, let S_j be the sort of A, B', a, b' and S_i be the original sort of A, B, a, b , we have $j \geq i$ if $B' > B$ or $b' > b$.

Lemma 2

LEMMA 2: According to the definition of Equation 12 by Selten, R. and T. Chmura (2008), for $\forall A, B, a, b \in R$, we have $S_l = \max[\min(A, a), \min(b, B)] = \text{min2}(A, B, a, b)$; min2 is the 2nd lowest value of the four matrix elements of the role (Up-Down or Right-Left)

Proof of theorem 1:

Here, we choose Player 1 for example. By lemma 2, when the sort of the four number is S_1 , we have the aspiration value (level) of Player 1 will be $AV=B=A_{22}$.

For convenience, we use the following payoff matrix:

$$\text{payoff matrix} = \begin{vmatrix} A_{11} & & A_{12} & \\ & B_{12} & & B_{22} \\ A_{21} & & A_{22} & \\ & B_{11} & & B_{21} \end{vmatrix}$$

We also can list all possible aspiration value of player 1 in Table 2.

[Insert Table 2 Here]

For any game with completely mixed equilibrium, we have $A > a$ ($A_{11} > A_{21}$). According to Table 2, from S_1 to S_2 , c_L^* keeps constant; from S_2 to S_3 , c_L^* increases by $1/2(A_{12} - A_{21})$, which is positive in S_2 ; from S_3 to S_4 , c_L^* increases to $A_{11} - A_{21}$ from $1/2(A_{11} - A_{21}) + 1/2(A_{12} - A_{21})$. Thus, we obtain that c_L^* is Monotone Increasing on S_i . Accordingly, d_D^* is Monotone Increasing, c_R^* and d_I^* Monotone Decreasing on S_i .

For any game with completely mixed equilibrium, we have $A > a$ ($A_{11} > A_{21}$). If a diamond Dummy is added on the payoff matrix, to player 1, we have r ($r > 0$) be added on B and b ($B > b$). By lemma 1, the sort of the four numbers in the Player 1's payoff matrix, S_i might be modified by adding the diamond Dummy. Thus, we obtain Theorem 1.

Table 2 Translated Matrix Element of the Sorted Sequences

			<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
		<i>Sequences Type</i>	S_1	S_2	S_3	S_4
		2^{nd} Min. Value	A_{22}	A_{21}	A_{12}	A_{11}
1	A_{11}^*	$1/2 (A_{11} - A_{22}) + A_{22}$	$1/2 (A_{12} - A_{21}) + A_{21}$	$1/2 (A_{11} - A_{12}) + A_{12}$	A_{11}	
2	A_{21}^*	$1/2 (A_{21} - A_{22}) + A_{22}$	A_{21}	A_{21}	A_{21}	
3	A_{22}^*	A_{22}	$1/2 (A_{22} - A_{21}) + A_{21}$	$1/2 (A_{22} - A_{12}) + A_{12}$	$1/2 (A_{22} - A_{11}) + A_{11}$	
4	A_{12}^*	A_{12}	A_{12}	A_{12}	$1/2 (A_{12} - A_{11}) + A_{11}$	
5	$c_L^* = A_{11}^* - A_{21}^*$	$1/2 (A_{11} - A_{21})$	$1/2 (A_{11} - A_{21})$	$1/2 (A_{11} - A_{21}) + 1/2 (A_{12} - A_{21})$	$A_{11} - A_{21}$	
6	$c_R^* = A_{22}^* - A_{12}^*$	$A_{22} - A_{12}$	$1/2 (A_{22} - A_{12}) + 1/2 (A_{21} - A_{12})$	$1/2 (A_{22} - A_{12})$	$1/2 (A_{22} - A_{12})$	
7	$d_D^* = B_{11}^* - B_{21}^*$	$1/2 (B_{11} - B_{21})$	$1/2 (B_{11} - B_{21})$	$1/2 (B_{11} - B_{21}) + 1/2 (B_{12} - B_{21})$	$B_{11} - B_{21}$	
8	$d_U^* = B_{22}^* - B_{12}^*$	$B_{22} - B_{12}$	$1/2 (B_{22} - B_{12}) + 1/2 (B_{21} - B_{12})$	$1/2 (B_{22} - B_{12})$	$1/2 (B_{22} - B_{12})$	