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Flows and Decompositions of Games: Harmonic and Potential Games

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Abstract

In this paper we introduce a novel flow representation for finite games in strategic form. This representation allows us to develop a canonical direct sum decomposition of an arbitrary game into three components, which we refer to as the *potential*, *harmonic* and *nonstrategic* components. We analyze natural classes of games that are induced by this decomposition, and in particular, focus on games with no harmonic component and games with no potential component. We show that the first class corresponds to the well-known *potential games*. We refer to the second class of games as *harmonic games*, and study the structural and equilibrium properties of this new class of games.

Intuitively, the potential component of a game captures interactions that can equivalently be represented as a common interest game, while the harmonic part represents the conflicts between the interests of the players. We make this intuition precise, by studying the properties of these two classes, and show that indeed they have quite distinct and remarkable characteristics. For instance, while finite potential games always have pure Nash equilibria, harmonic games generically never do. Moreover, we show that the nonstrategic component does not affect the equilibria of a game, but plays a fundamental role in their efficiency properties, thus decoupling the location of equilibria and their payoff-related properties. Exploiting the properties of the decomposition framework, we obtain explicit expressions for the projections of games onto the subspaces of potential and harmonic games. This enables an extension of the properties of potential and harmonic games to "nearby" games. We exemplify this point by showing that the set of approximate equilibria of an arbitrary game can be characterized through the equilibria of its projection onto the set of potential games.

Keywords: decomposition of games, potential games, harmonic games, strategic equivalence.

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1 Introduction

Potential games play an important role in game-theoretic analysis due to their desirable static properties (e.g., existence of a pure strategy Nash equilibrium) and tractable dynamics (e.g., convergence of simple user dynamics to a Nash equilibrium); see [32, 31, 35]. However, many multi-agent strategic interactions in economics and engineering cannot be modeled as a potential game.

This paper provides a novel flow representation of the preference structure in strategic-form finite games, which allows for delineating the fundamental characteristics in preferences that lead to potential games. This representation enables us to develop a canonical orthogonal decomposition of an arbitrary game into a potential component, a harmonic component, and a nonstrategic component, each with its distinct properties. The decomposition can be used to define the "distance" of an arbitrary game to the set of potential games. We use this fact to describe the approximate equilibria of the original game in terms of the equilibria of the closest potential game.

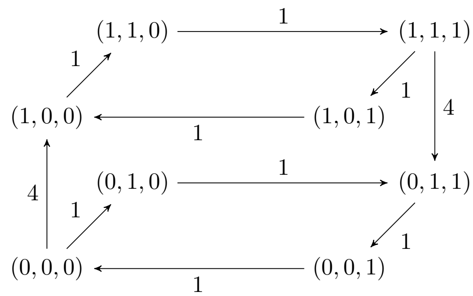
The starting point is to associate to a given finite game a *game graph*, where the set of nodes corresponds to the strategy profiles and the edges represent the "comparable strategy profiles" i.e., strategy profiles that differ in the strategy of a single player. The utility differences for the deviating players along the edges define a flow on the game graph. Although this graph contains strictly less information than the original description of the game in terms of utility functions, all relevant strategic aspects (e.g., equilibria) are captured.

Our first result provides a *canonical decomposition* of an arbitrary game using tools from the study of flows on graphs (which can be viewed as combinatorial analogues of the study of vector fields). In particular, we use the *Helmholtz decomposition theorem* (e.g., [21]), which enables the decomposition of a flow on a graph into three components: globally consistent, locally consistent (but globally inconsistent), and locally inconsistent component (see Theorem 3.1). The globally consistent component represents a gradient flow while the locally consistent flow corresponds to flows around global cycles. The locally inconsistent component represents local cycles (or circulations) around 3-cliques of the graph.

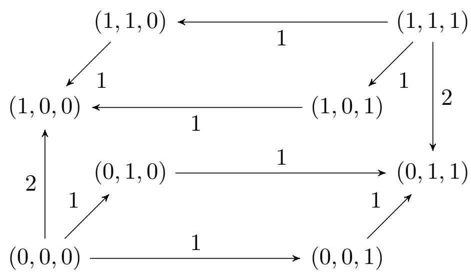
Our game decomposition has three components: *nonstrategic*, *potential* and *harmonic*. The first component represents the "nonstrategic interactions" in a game. Consider two games in which, given the strategies of the other players, each player's utility function differs by an additive constant. These two games have the same utility differences, and therefore they have the same flow representation. Moreover, since equilibria are defined in terms of utility differences, the two games have the same equilibrium set. We refer to such games as *strategically equivalent.* We normalize the utilities, and refer to the utility differences between a game and its normalization as the nonstrategic component of the game. Our next step is to remove the nonstrategic component and apply the Helmholtz decomposition to the remainder. The flow representation of a game defined in terms of utility functions (as opposed to preferences) does not exhibit local cycles, therefore the Helmholtz decomposition yields the two remaining components of a game: the *potential component* (gradient flow) and the *harmonic component* (global cycles). The decomposition result is particularly insightful for bimatrix games (i.e., finite games with two players, see Section 4.3), where the potential component represents the "team part" of the utilities (suitably perturbed to capture the utility matrix differences), and the harmonic component corresponds to a zero-sum game.

The canonical decomposition we introduce is illustrated in the following example.

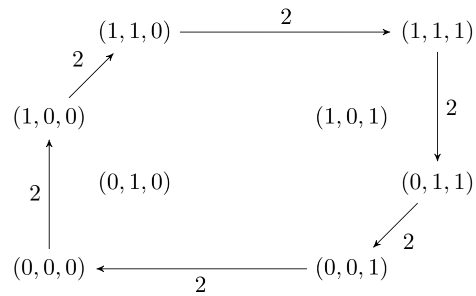
Example 1.1 (Road-sharing game)**.** *Consider a three-player game, where each player has to choose one of the two roads* $\{0,1\}$*. We denote the players by* d_1*,* d_2 *and* s*. The player* s *tries to avoid sharing the road with other players: its payoff decreases by* 2 *with each player* d_1 *and* d_2 *who shares the same road with it. The player* d_1 *receives a payoff* -1*, if* d_2 *shares the road with it and* 0



(a) Flow representation of the road-sharing game.



(b) Potential Component.



(c) Harmonic Component.

Figure 1: Potential-harmonic decomposition of the road-sharing game. An arrow between two strategy profiles, indicates the improvement direction in the payoff of the player who changes its strategy, and the associated number quantifies the improvement in its payoff.

otherwise. The payoff of d_2 *is equal to negative of the payoff of* d_1*, i.e.,* $u^{d_1} + u^{d_2} = 0$*. Intuitively, player* d_1 *tries to avoid player* d_2*, whereas player* d_2 *wants to use the same road with* d_1*.*

In Figure 1a we present the flow representation for this game (described in detail in Section 2.2), where the nonstrategic component has been removed. Figures 1b and 1c show the decomposition of this flow into its potential and harmonic components. In the figure, each tuple (a, b, c) *denotes a strategy profile, where player* s *uses strategy* a *and players* d_1 *and* d_2 *use strategies* b *and* c *respectively.*

These components induce a direct sum decomposition of the space of games into three respective subspaces, which we refer to as the *nonstrategic*, *potential* and *harmonic* subspaces, denoted by $\mathcal{N}$, $\mathcal{P}$, and $\mathcal{H}$, respectively. We use these subspaces to define classes of games with distinct equilibrium properties. We establish that the set of potential games coincides with the direct sum of the subspaces $\mathcal{P}$ and $\mathcal{N}$, i.e., potential games are those with no harmonic component. Similarly, we define a new class of games in which the potential component vanishes as *harmonic games*. The classical rock-paper-scissors and matching pennies games are examples of harmonic games. The decomposition then has the following structure:

$$\underbrace{\mathcal{P} \quad \oplus \quad \overbrace{\mathcal{N}}_{}}_{\text{Potential games}}\!\!\!\!\overbrace{\quad \oplus \quad \mathcal{H}}^{\text{Harmonic games}}.$$

Our second set of results establishes properties of potential and harmonic games and examines how the nonstrategic component of a game affects the efficiency of equilibria. Harmonic games can be characterized by the existence of improvement cycles, i.e., cycles in the game graph, where at each step the player that changes its action improves its payoffs. We show that harmonic games generically do not have pure Nash equilibria. Interestingly, for the special case when the number of strategies of each player is the same, a harmonic game satisfies a "multi-player zero-sum property" (i.e., the sum of utilities of all players is equal to zero at all strategy profiles). We also study the mixed Nash and correlated equilibria of harmonic games. We show that the uniformly mixed strategy profile (see Definition 5.2) is always a mixed Nash equilibrium and if there are two players in the game, the set of mixed Nash equilibria generically coincides with the set of correlated equilibria. We finally focus on the nonstrategic component of a game. As discussed above, the nonstrategic component does not affect the equilibrium set. Using this property, we show that by changing the nonstrategic component of a game, it is possible to make the set of Nash equilibria coincide with the set of Pareto optimal strategy profiles in a game.

Our third set of results focuses on the *projection of a game onto its respective components*. We first define a natural inner product and show that under this inner product the components in our decomposition are orthogonal. We further provide explicit expressions for the closest potential and harmonic games to a game with respect to the norm induced by the inner product. We use the distance of a game to its closest potential game to characterize the approximate equilibrium set in terms of the equilibria of the potential game.

The decomposition framework in this paper leads to the identification of subspaces of games with distinct and tractable equilibrium properties. Understanding the structural properties of these subspaces and the classes of games they induce, provides new insights and tools for analyzing the static and dynamical properties of general noncooperative games; further implications are outlined in Section 7.

Related literature Besides the works already mentioned, our paper is also related to several papers in the cooperative and noncooperative game theory literature:

- The idea of decomposing a game (using different approaches) into simpler games which admit more tractable equilibrium analysis appeared even in the early works in the cooperative game theory literature. In [45], the authors propose to decompose games with large number of players into games with fewer players. In [29, 13, 40], a different approach is followed: the authors identify cooperative games through the games' value functions (see [45]) and obtain decompositions of the value function into simpler functions. By defining the component games using the simpler value functions, they obtain decompositions of games. In this approach, the set of players is not made smaller or larger by the decomposition but the component games have simpler structure. Another method for decomposing the space of cooperative games appeared in [24, 26, 25]. In these papers, the algebraic properties of the space of games and the properties of the nullspace of the Shapley value operator (see [40]) and its orthogonal complement are exploited to decompose games. This approach does not necessarily simplify the analysis of games but it leads to an alternative expression for the Shapley value [25]. Our work is on decomposition of noncooperative games, and different from the above references since we explicitly exploit the properties of noncooperative games in our framework.

- In the context of noncooperative game theory, a decomposition for games in normal form appeared in [38]. In this paper, the author defines a component game for each subset of players and obtains a decomposition of normal form games with M players to 2^M component games. This method does not provide any insights about the properties of the component games, but yields alternative tests to check whether a game is a potential game or not.

We note that our decomposition approach is different than this work in the properties of the component games. In particular, using the global preference structure in games, our approach yields decomposition of games to three components with distinct equilibrium properties, and these properties can be exploited to gain insights about the static and dynamic features of the original game.

- Related ideas of representing finite strategic form games as graphs previously appeared in the literature to study different solution concepts in normal form games [14, 6]. In these references, the authors focus on the restriction of the game graph to best-reply paths and analyze the outcomes of games using this subgraph.

- In our work, the graph representation of games and the flows defined on this graph lead to a natural equivalence relation. Related notions of strategic equivalence are employed in the game theory literature to generalize the desirable static and dynamic properties of games to their equivalence classes [34, 37, 33, 46, 12, 16, 18, 23, 30, 17]. In [34], the authors refer to games which have the same better-response correspondence as equivalent games and study the equilibrium properties of games which are equivalent to zero-sum games. In [16, 18], the dynamic and static properties of certain classes of bimatrix games are generalized to their equivalence classes. Using the best-response correspondence instead of the better-response correspondence, the papers [37, 33, 46] define different equivalence classes of games. We note that the notion of strategic equivalence used in our paper implies some of the equivalence notions mentioned above. However, unlike these papers, our notion of strategic equivalence leads to a canonical decomposition of the space of games, which is then used to extend the desirable properties of potential games to "close" games that are not strategically equivalent.

- Despite the fact that harmonic games were not defined in the literature before (and thus, the term "harmonic" does not appear explicitly as such), specific instances of harmonic games were studied in different contexts. In [20], the authors study dynamics in "cyclic games" and obtain results about a class of harmonic games which generalize the matching pennies game. A parametrized version of Dawkins' battle of the sexes game, which is a harmonic game under certain conditions, is studied in [41]. Other examples of harmonic games have also appeared in the buyer/seller game of [9] and the crime deterrence game of [8].

Structure of the paper The remainder of this paper is organized as follows. In Section 2, we present the relevant game theoretic background and provide a representation of games in terms of graph flows. In Section 3, we state the Helmholtz decomposition theorem which provides the means of decomposing a flow into orthogonal components. In Section 4, we use this machinery to obtain a canonical decomposition of the space of games. We introduce in Section 5 natural classes of games, namely potential and harmonic games, which are induced by this decomposition and describe the equilibrium properties thereof. In Section 6, we define an inner product for the space of games, under which the components of games turn out to be orthogonal. Using this inner product and our decomposition framework we propose a method for projecting a given game to the spaces of potential and harmonic games. We then apply the projection to study the equilibrium properties of "near-potential" games. We close in Section 7 with concluding remarks and directions for future work.

2 Game-Theoretic Background

In this section, we describe the required game-theoretic background. Notation and basic definitions are given in Section 2.1. In Section 2.2, we provide an alternative representation of games in terms of flows on graphs. This representation is used in the rest of the paper to analyze finite games.

2.1 Preliminaries

A (noncooperative) *strategic-form finite game* consists of:

- A finite set of players, denoted $\mathcal{M} = \{1, \ldots, M\}$.
- Strategy spaces: A finite set of strategies (or actions) E^m, for every $m \in \mathcal{M}$. The joint strategy space is denoted by $E = \prod_{m \in \mathcal{M}} E^m$.
- Utility functions: $u^m : E \to \mathbb{R}$, $m \in \mathcal{M}$.

A (strategic-form) game instance is accordingly given by the tuple $\langle \mathcal{M}, \{E^m\}_{m \in \mathcal{M}}, \{u^m\}_{m \in \mathcal{M}} \rangle$, which for notational convenience will often be abbreviated to $\langle \mathcal{M}, \{E^m\}, \{u^m\} \rangle$.

We use the notation $\mathbf{p}^m \in E^m$ for a strategy of player m. A collection of players' strategies is given by $\mathbf{p} = \{\mathbf{p}^m\}_{m \in \mathcal{M}}$ and is referred to as a strategy profile. A collection of strategies for all players but the m-th one is denoted by $\mathbf{p}^{-m} \in E^{-m}$. We use $h_m = |E^m|$ for the cardinality of the strategy space of player m, and $|E| = \prod_{m=1}^{M} h_m$ for the overall cardinality of the strategy space. As an alternative representation, we shall sometimes enumerate the actions of the players, so that $E^m = \{1, \ldots, h_m\}$.

The basic solution concept in a noncooperative game is that of a *Nash Equilibrium* (NE). A (pure) Nash equilibrium is a strategy profile from which no player can unilaterally deviate and improve its payoff. Formally, a strategy profile $\mathbf{p} \triangleq \{\mathbf{p}^1, \ldots, \mathbf{p}^M\}$ is a Nash equilibrium if

$$u^m(\mathbf{p}^m, \mathbf{p}^{-m}) \geq u^m(\mathbf{q}^m, \mathbf{p}^{-m}), \quad \text{for every } \mathbf{q}^m \in E^m \text{ and } m \in \mathcal{M}. \tag{1}$$

To address strategy profiles that are approximately a Nash equilibrium, we introduce the concept of ϵ-equilibrium. A strategy profile $\mathbf{p} \triangleq \{\mathbf{p}^1, \ldots, \mathbf{p}^M\}$ is an ϵ-equilibrium if

$$u^m(\mathbf{p}^m, \mathbf{p}^{-m}) \geq u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - \epsilon \quad \text{for every } \mathbf{q}^m \in E^m \text{ and } m \in \mathcal{M}. \tag{2}$$

Note that a Nash equilibrium is an ϵ-equilibrium with $\epsilon = 0$.

The next lemma shows that the ϵ-equilibria of two games can be related in terms of the differences in utilities.

Lemma 2.1. *Consider two games $\mathcal{G}$ and $\hat{\mathcal{G}}$, which differ only in their utility functions, i.e., $\mathcal{G} = \langle \mathcal{M}, \{E^m\}, \{u^m\} \rangle$ and $\hat{\mathcal{G}} = \langle \mathcal{M}, \{E^m\}, \{\hat{u}^m\} \rangle$. Assume that $|u^m(\mathbf{p}) - \hat{u}^m(\mathbf{p})| \leq \epsilon_0$ for every $m \in \mathcal{M}$ and $\mathbf{p} \in E$. Then, every ϵ_1-equilibrium of $\hat{\mathcal{G}}$ is an ϵ-equilibrium of $\mathcal{G}$ for some $\epsilon \leq 2\epsilon_0 + \epsilon_1$ (and viceversa).*

Proof. Let $\mathbf{p}$ be an ϵ_1-equilibrium of $\hat{\mathcal{G}}$ and let $\mathbf{q} \in E$ be a strategy profile with $\mathbf{q}^k \neq \mathbf{p}^k$ for some $k \in \mathcal{M}$, and $\mathbf{q}^m = \mathbf{p}^m$ for every $m \in \mathcal{M} \setminus \{k\}$. Then,

$$u^k(\mathbf{q}) - u^k(\mathbf{p}) \leq u^k(\mathbf{q}) - u^k(\mathbf{p}) - (\hat{u}^k(\mathbf{q}) - \hat{u}^k(\mathbf{p})) + \epsilon_1 \leq 2\epsilon_0 + \epsilon_1,$$

where the first inequality follows since $\mathbf{p}$ is an ϵ_1-equilibrium of $\hat{\mathcal{G}}$, hence $\hat{u}^k(\mathbf{p}) - \hat{u}^k(\mathbf{q}) \geq -\epsilon_1$, and the second inequality follows by the lemma's assumption. $\square$

We turn now to describe a particular class of games that is central in this paper, the class of potential games [32].

Definition 2.1 (Potential Game)**.** *A potential game is a noncooperative game for which there exists a function $\phi : E \to \mathbb{R}$ satisfying*

$$\phi(\mathbf{p}^m, \mathbf{p}^{-m}) - \phi(\mathbf{q}^m, \mathbf{p}^{-m}) = u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m}), \tag{3}$$

for every $m \in \mathcal{M}$, $\mathbf{p}^m, \mathbf{q}^m \in E^m$, $\mathbf{p}^{-m} \in E^{-m}$. The function ϕ is referred to as a potential *function of the game.*

Potential games can be regarded as games in which the interests of the players are aligned with a global potential function ϕ. Games that obey condition (3) are also known in the literature as *exact* potential games, to distinguish them from other classes of games that relate to a potential function (in a different manner). For simplicity of exposition, we will often write 'potential games' when referring to exact potential games. Potential games have desirable equilibrium and dynamic properties as summarized in Section 5.1.

2.2 Games and Flows on Graphs

In noncooperative games, the utility functions capture the preferences of agents at each strategy profile. Specifically, the payoff difference $[u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m})]$ quantifies by how much player m prefers strategy $\mathbf{p}^m$ over strategy $\mathbf{q}^m$ (given that others play $\mathbf{p}^{-m}$). Note that a Nash equilibrium is defined in terms of payoff differences, suggesting that actual payoffs in the game are not required for the identification of equilibria, as long as the payoff differences are well defined.

A pair of strategy profiles that differ only in the strategy of a single player will be henceforth referred to as *comparable strategy profiles.* We denote the set (of pairs) of comparable strategy profiles by $A \subset E \times E$, i.e., $\mathbf{p}, \mathbf{q}$ are comparable if and only if $(\mathbf{p}, \mathbf{q}) \in A$. A pair of strategy profiles that differ only in the strategy of player m is called a pair of *m-comparable strategy profiles.* The set of pairs of m-comparable strategies is denoted by $A^m \subset E \times E$. Clearly, $\cup_m A^m = A$, where $A^m \cap A^k = \emptyset$ for any two different players m and k.

For any given m-comparable strategy profiles $\mathbf{p}$ and $\mathbf{q}$, the difference $[u^m(\mathbf{p}) - u^m(\mathbf{q})]$ would be henceforth identified as their *pairwise comparison.* For any game, we define the *pairwise comparison* function $X : E \times E \to \mathbb{R}$ as follows

$$X(\mathbf{p}, \mathbf{q}) = \begin{cases} u^m(\mathbf{q}) - u^m(\mathbf{p}) & \text{if } (\mathbf{p}, \mathbf{q}) \text{ are } m\text{-comparable for some } m \in \mathcal{M} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

In view of Definition 2.1, a game is an exact potential game if and only if there exists a function $\phi : E \to \mathbb{R}$ such that $\phi(\mathbf{q}) - \phi(\mathbf{p}) = X(\mathbf{p}, \mathbf{q})$ for any comparable strategy profiles $\mathbf{p}$ and $\mathbf{q}$. Note that the pairwise comparisons are uniquely defined for any given game. However, the converse is not true in the sense that there are infinitely many games that correspond to given pairwise comparisons. We exemplify this below.

Example 2.1. *Consider the payoff matrices of the two-player games in Tables 1a and 1b. For a given row and column, the first number denotes the payoff of the row player, and the second number denotes the payoff of the column player. The game in Table 1a is the "battle of the sexes" game, and the game in 1b is a variation in which the payoff of the row player is increased by* 1 *if the column player plays O.*

	O	F
O	3, 2	0, 0
F	0, 0	2, 3

(a) Battle of the sexes

	O	F
O	4, 2	0, 0
F	1, 0	2, 3

(b) Modified battle of the sexes

It is easy to see that these two games have the same pairwise comparisons, which will lead to identical equilibria for the two games: (O, O) *and* (F, F)*. It is only the actual equilibrium payoffs that would differ. In particular, in the equilibrium* (O, O)*, the payoff of the row player is increased by* 1*.*

The usual solution concepts in games (e.g., Nash, mixed Nash, correlated equilibria) are defined in terms of pairwise comparisons only. Games with identical pairwise comparisons share the same equilibrium sets. Thus, we refer to games with identical pairwise comparisons as *strategically equivalent games.*

By employing the notion of pairwise comparisons, we can concisely represent any strategic-form game in terms of a *flow* in a graph. We recall this notion next. Let $G = (N, L)$ be an undirected graph, with set of nodes N and set of links L. An *edge flow* (or just *flow*) on this graph is a function $Y : N \times N \to \mathbb{R}$ such that $Y(\mathbf{p}, \mathbf{q}) = -Y(\mathbf{q}, \mathbf{p})$ and $Y(\mathbf{p}, \mathbf{q}) = 0$ for $(\mathbf{p}, \mathbf{q}) \notin L$ [21, 2]. Note that the flow conservation equations are not enforced under this general definition.

Given a game $\mathcal{G}$, we define a graph where each node corresponds to a strategy profile, and each edge connects two comparable strategy profiles. This undirected graph is referred to as the *game graph* and is denoted by $G(\mathcal{G}) \triangleq (E, A)$, where E and A are the strategy profiles and pairs of comparable strategy profiles defined above, respectively. Notice that, by definition, the graph $G(\mathcal{G})$ has the structure of a direct product of M cliques (one per player), with clique m having h_m vertices. The pairwise comparison function $X : E \times E \to \mathbb{R}$ defines a flow on $G(\mathcal{G})$, as it satisfies $X(\mathbf{p}, \mathbf{q}) = -X(\mathbf{q}, \mathbf{p})$ and $X(\mathbf{p}, \mathbf{q}) = 0$ for $(\mathbf{p}, \mathbf{q}) \notin A$. This flow may thus serve as an equivalent representation of any game (up to a "non-strategic" component). It follows directly from the statements above that two games are strategically equivalent if and only if they have the same flow representation and game graph.

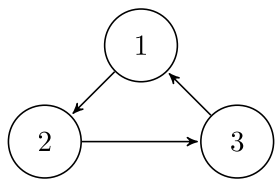
Two examples of game graph representations are given below.

Example 2.2. *Consider again the "battle of the sexes" game from Example 2.1. The game graph has four vertices, corresponding to the direct product of two 2-cliques, and is presented in Figure 2.*

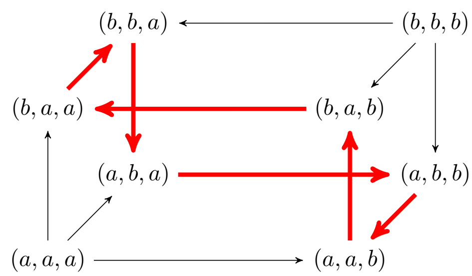
$$\begin{tikzcd}
(O,O) & (O,F) \arrow[l, swap, "2"] \arrow[d, "2"] \\
(F,O) \arrow[u, "3"] \arrow[r, "3"] & (F,F)
\end{tikzcd}$$

Figure 2: Flows on the game graph corresponding to "battle of the sexes" (Example 2.2).

Example 2.3. *Consider a three-player game, where each player can choose between two strategies* $\{a, b\}$*. We represent the strategic interactions among the players by the directed graph in Figure 3a, where the payoff of player* i *is* -1 *if its strategy is identical to the strategy of its successor* *(indexed $[i \mod 3+1]$), and 1 otherwise. Figure 3b depicts the associated game graph and pairwise comparisons of this game, where the arrow direction corresponds to an increase in the utility by the deviating player. The numerical values of the flow are omitted from the figure, and are all equal to 2; thus notice that flow conservation does not hold. The highlighted cycle will play an important role later, after we discuss potential games.*



(a) Player Interaction Graph



(b) Flows on the game graph

Figure 3: A three-player game, and associated flow on its game graph. Each arrow designates an improvement in the payoff of the agent who unilaterally modifies its strategy. The highlighted cycle implies that in this game, there can be an infinitely long sequence of profitable unilateral deviations.

The representation of a game as a flow in a graph is natural and useful for the understanding of its strategic interactions, as it abstracts away the absolute utility values and allows for more direct equilibrium-related interpretation. In more mathematical terms, it considers the quotient of the utilities modulo the subspace of games that are "equivalent" to the trivial game (the game where all players receive zero payoff at all strategy profiles), and allows for the identification of "equivalent" games as the same object, a point explored in more detail in later sections. The game graph also contains much structural information. For example, the highlighted sequence of arrows in Figure 3b forms a directed cycle, indicating that no strategy profile within that cycle could be a pure Nash equilibrium. Our goal in this paper is to use tools from the theory of graph flows to decompose a game into components, each of which admits tractable equilibrium characterization. The next section provides an overview of the tools that are required for this objective.

3 Flows and Helmholtz Decomposition

The objective of this section is to provide a brief overview of the notation and tools required for the analysis of flows on graphs. The basic high-level idea is that under certain conditions (e.g., for graphs arising from games), it is possible to consider graphs as natural topological spaces with nontrivial homological properties. These topological features (e.g., the presence of "holes", due to the presence of different players) in turn enable the possibility of interesting flow decompositions. In what follows, we make these ideas precise. For simplicity and accessibility to a wider audience, we describe the methods in relatively elementary linear algebraic language, limiting the usage of algebraic topology notions whenever possible. The main technical tool we use is the Helmholtz decomposition theorem, a classical result from algebraic topology with many applications in applied mathematics, including among others electromagnetism, computational geometry and data visualization; see e.g. [36, 42]. In particular, we mention the very interesting recent work by Jiang et al.

[21], where the Helmholtz/Hodge decomposition is applied to the problem of statistical ranking for sets of incomplete data.

Consider an undirected graph $G = (E, A)$, where E is the set of the nodes, and A is the set of edges of the graph[1]. Since the graph is undirected $(\mathbf{p}, \mathbf{q}) \in A$ if and only if $(\mathbf{q}, \mathbf{p}) \in A$. We denote the set of 3-*cliques* of the graph by $T = \{(\mathbf{p}, \mathbf{q}, \mathbf{r}) | (\mathbf{p}, \mathbf{q}), (\mathbf{q}, \mathbf{r}), (\mathbf{p}, \mathbf{r}) \in A\}$.

We denote by $C_0 = \{f | \ f : E \to \mathbb{R}\}$ the set of real-valued functions on the set of nodes. Recall that the *edge flows* $X : E \times E \to \mathbb{R}$ are functions which satisfy

$$X(\mathbf{p}, \mathbf{q}) = \begin{cases} -X(\mathbf{q}, \mathbf{p}) & \text{if } (\mathbf{p}, \mathbf{q}) \in A \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Similarly the *triangular flows* $\Psi : E \times E \times E \to \mathbb{R}$ are functions for which

$$\Psi(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \Psi(\mathbf{q}, \mathbf{r}, \mathbf{p}) = \Psi(\mathbf{r}, \mathbf{p}, \mathbf{q}) = -\Psi(\mathbf{q}, \mathbf{p}, \mathbf{r}) = -\Psi(\mathbf{p}, \mathbf{r}, \mathbf{q}) = -\Psi(\mathbf{r}, \mathbf{q}, \mathbf{p}), \tag{6}$$

and $\Psi(\mathbf{p}, \mathbf{q}, \mathbf{r}) = 0$ if $(\mathbf{p}, \mathbf{q}, \mathbf{r}) \notin T$. Given a graph G, we denote the set of all possible edge flows by C_1 and the set of triangular flows by C_2. Notice that both C_1 and C_2 are alternating functions of their arguments. It follows from (5) that $X(\mathbf{p}, \mathbf{p}) = 0$ for all $X \in C_1$.

The sets C_0, C_1 and C_2 have a natural structure of vector spaces, with the obvious operations of addition and scalar multiplication. In this paper, we use the following inner products:

$$\begin{aligned} \langle \phi_1, \phi_2 \rangle_0 &= \sum_{\mathbf{p} \in E} \phi_1(\mathbf{p}) \phi_2(\mathbf{p}). \\ \langle X, Y \rangle_1 &= \frac{1}{2} \sum_{(\mathbf{p}, \mathbf{q}) \in A} X(\mathbf{p}, \mathbf{q}) Y(\mathbf{p}, \mathbf{q}) \\ \langle \Psi_1, \Psi_2 \rangle_2 &= \sum_{(\mathbf{p}, \mathbf{q}, \mathbf{r}) \in T} \Psi_1(\mathbf{p}, \mathbf{q}, \mathbf{r}) \Psi_2(\mathbf{p}, \mathbf{q}, \mathbf{r}). \end{aligned} \tag{7}$$

We shall frequently drop the subscript in the inner product notation, as the respective space will often be clear from the context.

We next define linear operators that relate the above defined objects. To that end, let $W : E \times E \to \mathbb{R}$ be an indicator function for the edges of the graph, namely

$$W(\mathbf{p}, \mathbf{q}) = \begin{cases} 1 & \text{if } (\mathbf{p}, \mathbf{q}) \in A \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Notice that $W(\mathbf{p}, \mathbf{q})$ can be simply interpreted as the adjacency matrix of the graph G.

The first operator of interest is the *combinatorial gradient operator* $\delta_0 : C_0 \to C_1$, given by

$$(\delta_0 \phi)(\mathbf{p}, \mathbf{q}) = W(\mathbf{p}, \mathbf{q})(\phi(\mathbf{q}) - \phi(\mathbf{p})), \quad \mathbf{p}, \mathbf{q} \in E, \tag{9}$$

for $\phi \in C_0$. An operator which is used in the characterization of "circulations" in edge flows is the *curl operator* $\delta_1 : C_1 \to C_2$, which is defined for all $X \in C_1$ and $\mathbf{p}, \mathbf{q}, \mathbf{r} \in E$ as

$$(\delta_1 X)(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \begin{cases} X(\mathbf{p}, \mathbf{q}) + X(\mathbf{q}, \mathbf{r}) + X(\mathbf{r}, \mathbf{p}) & \text{if } (\mathbf{p}, \mathbf{q}, \mathbf{r}) \in T, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

[1] The results discussed in this section apply to arbitrary graphs. We use the notation introduced in Section 2 since in the rest of the paper we focus on the game graph introduced there.

We denote the adjoints of the operators δ_0 and δ_1 by δ_0^* and δ_1^* respectively. Recall that given inner products $\langle \cdot, \cdot \rangle_k$ on C_k, the adjoint of δ_k, namely $\delta_k^* : C_{k+1} \to C_k$, is the unique linear operator satisfying

$$\langle \delta_k f_k, g_{k+1} \rangle_{k+1} = \langle f_k, \delta_k^* g_{k+1} \rangle_k, \tag{11}$$

for all $f_k \in C_k$, $g_{k+1} \in C_{k+1}$.

Using the definitions in (11), (9) and (7), it can be readily seen that the adjoint $\delta_0^* : C_1 \to C_0$ of the combinatorial gradient δ_0 satisfies

$$(\delta_0^* X)(\mathbf{p}) = -\sum_{\mathbf{q}|(\mathbf{p},\mathbf{q}) \in A} X(\mathbf{p}, \mathbf{q}) = -\sum_{\mathbf{q} \in E} W(\mathbf{p}, \mathbf{q}) X(\mathbf{p}, \mathbf{q}). \tag{12}$$

Note that $-(\delta_0^* X)(\mathbf{p})$ represents the total flow "leaving" $\mathbf{p}$. We shall sometimes refer to the operator $-\delta_0^*$ as the *divergence* operator, due to its similarity to the divergence operator in Calculus.

The domains and codomains of the operators $\delta_0, \delta_1, \delta_0^*, \delta_1^*$ are summarized below.

$$\begin{aligned} &C_0 \xrightarrow{\delta_0} C_1 \xrightarrow{\delta_1} C_2 \\ &C_0 \xleftarrow{\delta_0^*} C_1 \xleftarrow{\delta_1^*} C_2. \end{aligned} \tag{13}$$

We next define the Laplacian operator, $\Delta_0 : C_0 \to C_0$, given by

$$\Delta_0 \triangleq \delta_0^* \circ \delta_0, \tag{14}$$

where $\circ$ represents operator composition. To simplify the notation, we henceforth omit $\circ$ and write $\Delta_0 = \delta_0^* \delta_0$. Note that functions in C_0 can be represented by vectors of length $|E|$ by indexing all nodes of the graph and constructing a vector whose ith entry is the function evaluated at the ith node. This allows us to easily represent these operators in terms of matrices. In particular, the Laplacian can be expressed as a square matrix of size $|E| \times |E|$; using the definitions for δ_0 and δ_0^*, it follows that

$$[\Delta_0]_{\mathbf{p},\mathbf{q}} = \begin{cases} \sum_{\mathbf{r} \in E} W(\mathbf{p}, \mathbf{r}) & \text{if } \mathbf{p} = \mathbf{q} \\ -1 & \text{if } \mathbf{p} \neq \mathbf{q} \text{ and } (\mathbf{p}, \mathbf{q}) \in A \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

where, with some abuse of the notation, $[\Delta_0]_{\mathbf{p},\mathbf{q}}$ denotes the entry of the matrix Δ_0, with rows and columns indexed by the nodes $\mathbf{p}$ and $\mathbf{q}$. The above matrix naturally coincides with the definition of a Laplacian of an undirected graph [7].

Since the entry of $\Delta_0 \phi$ corresponding to $\mathbf{p}$ equals $\sum_{\mathbf{q}} W(\mathbf{p}, \mathbf{q}) \big(\phi(\mathbf{p}) - \phi(\mathbf{q})\big)$, the Laplacian operator gives a measure of the aggregate "value" of a node over all its neighbors. A related operator is

$$\Delta_1 \triangleq \delta_1^* \circ \delta_1 + \delta_0 \circ \delta_0^*, \tag{16}$$

known in the literature as the vector Laplacian [21].

We next provide additional flow-related terminology which will be used in association with the above defined operators, and highlight some of their basic properties. In analogy to the well-known identity in vector calculus, curl $\circ$ grad $= 0$, we have that δ_0 is a closed form, i.e., $\delta_1 \circ \delta_0 = 0$. An edge flow $X \in C_1$ is said to be *globally consistent* if X corresponds to the combinatorial gradient of some $f \in C_0$, i.e., $X = \delta_0 f$; the function f is referred to as the *potential function* corresponding to X. Equivalently, the set of globally consistent edge flows can be represented as the image of the gradient operator, namely im (δ_0). By the closedness of δ_0, observe that $\delta_1 X = 0$

for every globally consistent edge flow X. We define *locally consistent* edge flows as those satisfying $(\delta_1 X)(\mathbf{p},\mathbf{q},\mathbf{r}) = X(\mathbf{p},\mathbf{q}) + X(\mathbf{q},\mathbf{r}) + X(\mathbf{r},\mathbf{p}) = 0$ for all $(\mathbf{p},\mathbf{q},\mathbf{r}) \in T$. Note that the kernel of the curl operator $\ker(\delta_1)$ is the set of locally consistent edge flows. The latter subset is generally not equivalent to im (δ_0), as there may exist edge flows that are globally inconsistent but locally consistent (in fact, this will happen whenever the graph has a nontrivial topology). We refer to such flows as *harmonic flows*. Note that the operators δ_0, δ_1 are linear operators, thus their image spaces are orthogonal to the kernels of their adjoints, i.e., im $(\delta_0) \perp \ker(\delta_0^*)$ and im $(\delta_1) \perp \ker(\delta_1^*)$ [similarly, im $(\delta_0^*) \perp \ker(\delta_0)$ and im $(\delta_1^*) \perp \ker(\delta_1)$ as can be easily verified using (11)].

We state below a basic flow-decomposition theorem, known as the Helmholtz Decomposition[2], which will be used in our context of noncooperative games. The theorem implies that any graph flow can be decomposed into three orthogonal flows.

Theorem 3.1 (Helmholtz Decomposition). *The vector space of edge flows C_1 admits an orthogonal decomposition*

$$C_1 = \text{im } (\delta_0) \oplus \ker(\Delta_1) \oplus \text{im } (\delta_1^*), \tag{17}$$

where $\ker(\Delta_1) = \ker(\delta_1) \cap \ker(\delta_0^*)$.

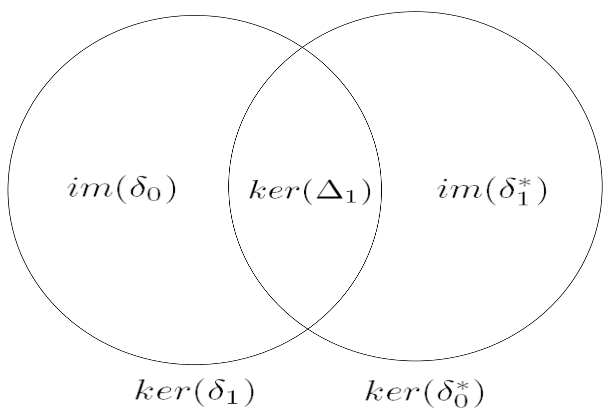


Figure 4: Helmholtz decomposition of C_1

Below we summarize the interpretation of each of the components in the Helmholtz decomposition (see also Figure 4):

- im (δ_0) – globally consistent flows.
- $\ker(\Delta_1) = \ker(\delta_1) \cap \ker(\delta_0^*)$ – harmonic flows, which are globally inconsistent but locally consistent. Observe that $\ker(\delta_1)$ consists of locally consistent flows (that may or may not be globally consistent), while $\ker(\delta_0^*)$ consists of globally inconsistent flows (that may or may not be locally consistent).
- im (δ_1^*) (or equivalently, the orthogonal complement of $\ker(\delta_1)$) – locally inconsistent flows.

We conclude this section with a brief remark on the decomposition and the flow conservation. For $X \in C_1$, if $\delta_0^* X = 0$, i.e., if for every node, the total flow leaving the node is zero, then we say

[2]The Helmholtz Decomposition can be generalized to higher dimensions through the Hodge Decomposition theorem (see [21]), however this generalization is not required for our purposes.

that X satisfies the flow conservation condition. Theorem 3.1 implies that X satisfies this condition only when $X \in \ker(\delta_0^*) = \text{im } (\delta_0)^\perp = \ker(\Delta_1) \oplus \text{im } (\delta_1^*)$. Thus, the flow conservation condition is satisfied for harmonic flows and locally inconsistent flows but not for globally consistent flows.

4 Canonical Decomposition of Games

In this section we obtain a canonical decomposition of an arbitrary game into basic components, by combining the game graph representation introduced in Section 2.2 with the Helmholtz decomposition discussed above.

Section 4.1 introduces the relevant operators that are required for formulating the results. In Section 4.2 we provide the basic decomposition theorem, which states that the space of games can be decomposed as a direct sum of three subspaces, referred to as the *potential*, *harmonic* and *nonstrategic* subspaces. In Section 4.3, we focus on bimatrix games, and provide explicit expressions for the decomposition.

4.1 Preliminaries

We consider a game $\mathcal{G}$ with set of players $\mathcal{M}$, strategy profiles $E \triangleq E^1 \times \cdots \times E^M$, and game graph $G(\mathcal{G}) = (E, A)$. Using the notation of the previous section, the utility functions of each player can be viewed as elements of C_0, i.e., $u^m \in C_0$ for all $m \in \mathcal{M}$. For given $\mathcal{M}$ and E, every game is uniquely defined by its set of utility functions. Hence, the space of games with players $\mathcal{M}$ and strategy profiles E can be identified as $\mathcal{G}_{\mathcal{M},E} \cong C_0^M$. In the rest of the paper we use the notations $\{u^m\}_{m\in\mathcal{M}}$ and $\mathcal{G} = \langle \mathcal{M}, \{E^m\}, \{u^m\}\rangle$ interchangeably when referring to games.

The pairwise comparison function $X(\mathbf{p}, \mathbf{q})$ of a game, defined in (4), corresponds to a flow on the game graph, and hence it belongs to C_1. In general, the flows representing games have some special structure. For example, the pairwise comparison between any two comparable strategy profiles is associated with the payoff of exactly a single player. It is therefore required to introduce *player-specific* operators and highlight some important identities between them, as we elaborate below.

Let $W^m : E \times E \to \mathbb{R}$ be the indicator function for m-comparable strategy profiles, namely

$$W^m(\mathbf{p}, \mathbf{q}) = \begin{cases} 1 & \text{if } \mathbf{p}, \mathbf{q} \text{ are } m\text{-comparable} \\ 0 & \text{otherwise.} \end{cases}$$

Recalling that any pair of strategy profiles cannot be comparable by more than a single user, we have

$$W^m(\mathbf{p}, \mathbf{q}) W^k(\mathbf{p}, \mathbf{q}) = 0, \quad \text{for all } k \neq m \text{ and } \mathbf{p}, \mathbf{q} \in E, \tag{18}$$

and

$$W = \sum_{m\in\mathcal{M}} W^m, \tag{19}$$

where W is the indicator function of comparable strategy profiles (edges of the game graph) defined in (8). Note that this can be interpreted as a decomposition of the adjacency matrix of $G(\mathcal{G})$, where the different components correspond to the edges associated with different players.

Given $\phi \in C_0$, we define $D_m : C_0 \to C_1$ such that

$$(D_m\phi)(\mathbf{p}, \mathbf{q}) = W^m(\mathbf{p}, \mathbf{q}) \left(\phi(\mathbf{q}) - \phi(\mathbf{p})\right). \tag{20}$$

This operator quantifies the change in ϕ between strategy profiles that are m-comparable. Using this operator, we can represent the pairwise differences X of a game with payoffs $\{u^m\}_{m\in\mathcal{M}}$ as follows:

$$X = \sum_{m\in\mathcal{M}} D_m u^m. \tag{21}$$

We define a relevant operator $D : C_0^M \to C_1$, such that $D = [D_1 \dots, D_M]$. As can be seen from (21), for a game with collection of utilities $u = [u^1; u^2 \dots; u^M] \in C_0^M$, the pairwise differences can alternatively be represented by Du.

Let $\Lambda_m : C_1 \to C_1$ be a scaling operator so that

$$(\Lambda_m X)(\mathbf{p}, \mathbf{q}) = W^m(\mathbf{p}, \mathbf{q}) X(\mathbf{p}, \mathbf{q})$$

for every $X \in C_1$, $\mathbf{p}, \mathbf{q} \in E$. From (19), it can be seen that for any $X \in C_1$, $\sum_m \Lambda_m X = X$. The definition of Λ_m and (18) imply that $\Lambda_m \Lambda_k = 0$ for $k \neq m$. Additionally, the definition of the inner product in C_1 implies that for $X, Y \in C_1$, it follows that $\langle \Lambda_m X, Y\rangle = \langle X, \Lambda_m Y\rangle$, i.e., Λ_m is self-adjoint.

This operator provides a convenient description for the operator D_m. From the definitions of D_m and Λ_m, it immediately follows that $D_m = \Lambda_m \delta_0$, and since $\sum_m \Lambda_m X = X$ for all $X \in C_1$,

$$\delta_0 = \sum_m \Lambda_m \delta_0 = \sum_m D_m.$$

Since Λ_m is self-adjoint, the adjoint of D_m, which is denoted by $D_m^* : C_1 \to C_0$, is given by:

$$D_m^* = \delta_0^* \Lambda_m.$$

Using (12) and the above definitions, it follows that

$$(D_m^* X)(\mathbf{p}) = -\sum_{\mathbf{q}\in E} W^m(\mathbf{p}, \mathbf{q}) X(\mathbf{p}, \mathbf{q}), \qquad \text{for all } X \in C_1, \tag{22}$$

and

$$\delta_0^* = \sum_{m\in\mathcal{M}} D_m^*. \tag{23}$$

Observe that $D_k^* D_m = \delta_0^* \Lambda_k \Lambda_m \delta_0 = 0$ for $k \neq m$. This immediately implies that the image spaces of $\{D_m\}_{m\in\mathcal{M}}$ are orthogonal, i.e., $D_k^* D_m = 0$. Let $D_m^\dagger$ denote the (Moore-Penrose) pseudoinverse of D_m, with respect to the inner products introduced in Section 3. By the properties of the pseudoinverse, we have $\ker D_m^\dagger = (\text{im } D_m)^\perp$. Thus, orthogonality of the image spaces of D_k operators imply that $D_k^\dagger D_m = 0$ for $k \neq m$.

The orthogonality leads to the following expression for the Laplacian operator,

$$\Delta_0 = \delta_0^* \delta_0 = \sum_{k\in\mathcal{M}} D_k^* \sum_{m\in\mathcal{M}} D_m = \sum_{m\in\mathcal{M}} D_m^* D_m. \tag{24}$$

In view of (20) and (22), D_m and $-D_m^*$ are the gradient and divergence operators on the graph of m-comparable strategy profiles (E, A^m). Therefore, the operator $\Delta_{0,m} \triangleq D_m^* D_m$ is the Laplacian of the graph induced by m-comparable strategies, and is referred to as the *Laplacian operator of the m-comparable strategy profiles.* It follows from (24) that

$$\Delta_0 = \sum_{m\in\mathcal{M}} \Delta_{0,m}.$$

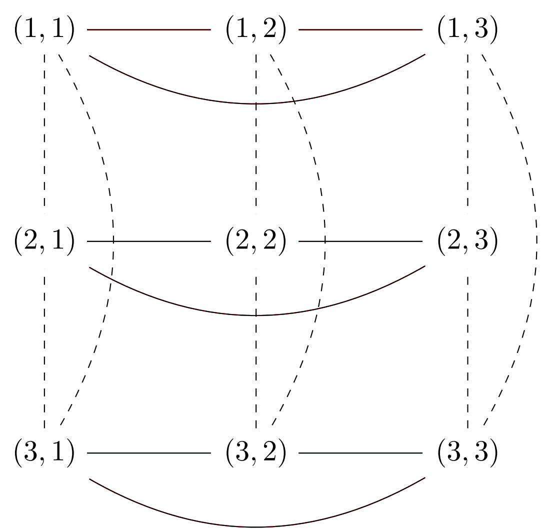


Figure 5: A game with two players, each of which has three strategies. A node (i, j) represents a strategy profile in which player 1 and player 2 use strategies i and j, respectively. The Laplacian $\Delta_{0,1}$ ($\Delta_{0,2}$) is defined on the graph whose edges are represented by dashed (solid) lines. The Laplacian Δ_0 is defined on the graph that includes all edges.

The relation between the Laplacian operators Δ_0 and $\Delta_{0,m}$ is illustrated in Figure 5.

Similarly, $\delta_1\Lambda_m$ is the curl operator associated with the subgraph (E, A^m). From the closedness of the curl ($\delta_1\Lambda_m$) and gradient ($\Lambda_m\delta_0$) operators defined on this subgraph, we obtain $\delta_1\Lambda_m^2\delta_0 = 0$. Observing that $\Lambda_m^2\delta_0 = \Lambda_m\delta_0 = D_m$, it follows that

$$\delta_1 D_m = 0. \tag{25}$$

This result also implies that $\delta_1 D = 0$, i.e., the pairwise comparisons of games belong to $\ker\delta_1$. Thus, it follows from Theorem 3.1 that the pairwise comparisons do not have a locally inconsistent component.

Lastly, we introduce projection operators that will be useful in the subsequent analysis. Consider the operator,

$$\Pi_m = D_m^\dagger D_m.$$

For any linear operator L, $L^\dagger L$ is a projection operator on the orthogonal complement of the kernel of L (see [15]). Since D_m is a linear operator, Π_m is a projection operator to the orthogonal complement of the kernel of D_m. Using these operators, we define $\Pi : C_0^M \to C_0^M$ such that $\Pi = diag(\Pi_1, \dots, \Pi_M)$, i.e., for $u = \{u_m\}_{m\in\mathcal{M}} \in C_0^M$, we have $\Pi u = [\Pi_1 u^1; \dots \Pi_M u^M] \in C_0^M$. We extend the inner product in C_0 to C_0^M (by defining the inner product as the sum of the inner products in all C_0 components), and denote by $D^\dagger$ the pseudoinverse of D according to this inner product. In Lemma 4.4, we will show that Π is equivalent to the projection operator to the orthogonal complement of the kernel of D, i.e., $\Pi = D^\dagger D$.

For easy reference, Table 1 provides a summary of notation. We next state some basic facts about the operators we introduced, which will be used in the subsequent analysis. The proofs of these results can be found in Appendix A.

$\mathcal{G}$	A game instance $\langle \mathcal{M}, \{E^m\}_{m\in\mathcal{M}}, \{u^m\}_{m\in\mathcal{M}}\rangle$.
$\mathcal{M}$	Set of players, $\{1,\ldots,M\}$.
E^m	Set of actions for player m, $E^m = \{1,\ldots,h_m\}$.
E	Joint action space $\prod_{m\in\mathcal{M}} E^m$.
u^m	Utility function of player m. We have $u^m \in C_0$.
W^m	Indicator function for m-comparable strategy profiles, $W^m : E\times E \to \{0,1\}$.
W	A function indicating whether strategy profiles are comparable, $W : E\times E \to \{0,1\}$.
C_0	Space of utilities, $C_0 = \{u^m \mid u^m : E \to \mathbb{R}\}$. Note that $C_0 \cong \mathbb{R}^{\lvert E\rvert}$.
C_1	Space of pairwise comparison functions from $E\times E$ to $\mathbb{R}$.
δ_0	Gradient operator, $\delta_0 : C_0 \to C_1$, satisfying $(\delta_0\phi)(\mathbf{p},\mathbf{q}) = W(\mathbf{p},\mathbf{q})\,(\phi(\mathbf{q}) - \phi(\mathbf{p}))$.
D_m	$D_m : C_0 \to C_1$, such that $(D_m\phi)(\mathbf{p},\mathbf{q}) = W^m(\mathbf{p},\mathbf{q})\,(\phi(\mathbf{q}) - \phi(\mathbf{p}))$.
D	$D : C_0^M \to C_1$, such that $D(u^1;\ldots;u^M) = \sum_m D_m u^m$.
δ_0^*, D_m^*	$\delta_0^*, D_m^* : C_1 \to C_0$ are the adjoints of the operators δ_0 and D_m, respectively.
Δ_0	Laplacian for the game graph. $\Delta_0 : C_0 \to C_0$; satisfies $\Delta_0 = \delta_0^*\delta_0 = \sum_m \Delta_{0,m}$.
$\Delta_{0,m}$	Laplacian for the graph of m-comparable strategies, $\Delta_{0,m} : C_0 \to C_0$; satisfies $\Delta_{0,m} = D_m^* D_m = D_m^*\delta_0$.
Π_m	Projection operator onto the orthogonal complement of kernel of D_m, $\Pi_m : C_0 \to C_0$; satisfies $\Pi_m = D_m^\dagger D_m$.

Table 1: Notation summary

Lemma 4.1. *The Laplacian of the graph induced by m-comparable strategies and the projection operator Π_m are related by $\Delta_{0,m} = h_m \Pi_m$, where $h_m = \lvert E^m\rvert$ denotes the number of strategies of player m.*

Lemma 4.2. *The kernels of operators D_m, Π_m and $\Delta_{0,m}$ coincide, namely* $\ker(D_m) = \ker(\Pi_m) = \ker(\Delta_{0,m})$. *Furthermore, a basis for these kernels is given by a collection $\{\nu_{\mathbf{q}^{-m}}\}_{\mathbf{q}^{-m}\in E^{-m}} \in C_0$ such that*

$$\nu_{\mathbf{q}^{-m}}(\mathbf{p}) = \begin{cases} 1 & \textit{if } \mathbf{p}^{-m} = \mathbf{q}^{-m} \\ 0 & \textit{otherwise} \end{cases} \tag{26}$$

Lemma 4.3. *The Laplacian Δ_0 of the game graph (the graph of comparable strategy profiles) always has eigenvalue 0 with multiplicity 1, corresponding to the constant eigenfunction (i.e., $f \in C_0$ such that $f(\mathbf{p}) = 1$ for all $\mathbf{p} \in E$).*

Lemma 4.4. *The pseudoinverses of operators D_m and D satisfy the following identities: (i) $D_m^\dagger = \frac{1}{h_m} D_m^*$, (ii) $(\sum_i D_i)^\dagger D_j = (\sum_i D_i^* D_i)^\dagger D_j^* D_j$, (iii) $D^\dagger = [D_1^\dagger;\ldots;D_M^\dagger]$, (iv) $\Pi = D^\dagger D$ (v) $DD^\dagger\delta_0 = \delta_0$.*

4.2 Decomposition of Games

In this subsection we prove that the space of games $\mathcal{G}_{\mathcal{M},E}$ is a direct sum of three subspaces – potential, harmonic and nonstrategic, each with distinguishing properties.

We start our discussion by formalizing the notion of nonstrategic information. Consider two games $\mathcal{G}$, $\hat{\mathcal{G}} \in \mathcal{G}_{\mathcal{M},E}$ with utilities $\{u^m\}_{m\in\mathcal{M}}$ and $\{\hat{u}^m\}_{m\in\mathcal{M}}$ respectively. Assume that the utility functions $\{u^m\}_{m\in\mathcal{M}}$ satisfy $u^m(\mathbf{p}^m,\mathbf{p}^{-m}) = \hat{u}^m(\mathbf{p}^m,\mathbf{p}^{-m}) + \alpha(\mathbf{p}^{-m})$ where α is an arbitrary function. It can be readily seen that these two games have exactly the same pairwise comparison functions, hence they are strategically equivalent. To express the same idea in words, whenever we add to the utility of one player an arbitrary function of the actions of the others, this does not directly affect the incentives of the player to choose among his/her possible actions. Thus, pairwise comparisons of utilities (or equivalently, the game graph representation) uniquely identify equivalent classes of games that have identical properties in terms of, for instance, sets of equilibria[3]. To fix a representative for each strategically equivalent game, we introduce below a notion of games where the nonstrategic information has been removed.

Definition 4.1 (Normalized games)**.** *We say that a game with utility functions* $\{u^m\}_{m\in\mathcal{M}}$ *is* normalized *or does not contain nonstrategic information if*

$$\sum_{\mathbf{p}^m} u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = 0 \tag{27}$$

for all $\mathbf{p}^{-m} \in E^{-m}$ *and all* $m \in \mathcal{M}$*.*

Note that removing the nonstrategic information amounts to normalizing the sum of the payoffs in the game. Normalization can be made with an arbitrary constant. However, in order to simplify the subsequent analysis we normalize the sum of the payoffs to zero. Intuitively, this suggests that given the strategies of an agent's opponents, the average payoff its strategies yield, is equal to zero. The following lemma characterizes the set of normalized games in terms of the operators introduced in the previous section.

Lemma 4.5. *Given a game* $\mathcal{G}$ *with utilities* $u = \{u^m\}_{m\in\mathcal{M}}$*, the following are equivalent: (i)* $\mathcal{G}$ *is normalized, (ii)* $\Pi_m u^m = u^m$ *for all* m*, (iii)* $\Pi u = u$*, (iv)* $u \in (\ker D)^\perp$*.*

Proof. The equivalence of (iii) and (iv) is immediate since by Lemma 4.4, $\Pi = D^\dagger D$ is a projection operator to the orthogonal complement of the kernel of D. The equivalence of (ii) and (iii) follows from the definition of $\Pi = diag(\Pi_1, \dots, \Pi_M)$. To complete the proof we prove (i) and (ii) are equivalent.

Observe that (27) holds if and only if $\langle u^m, \nu_{\mathbf{q}^{-m}}(\mathbf{p})\rangle = 0$ for all $\mathbf{q}^{-m} \in E^{-m}$, where $\nu_{\mathbf{q}^{-m}}$ is as defined in (26). Lemma 4.2 implies that $\{\nu_{\mathbf{q}^{-m}}\}$ are basis vectors of $\ker D_m$. Thus, it follows that (27) holds if and only if u^m is orthogonal to all of the basis vectors of $\ker D_m$, or equivalently when $u^m \in (\ker D_m)^\perp$. Since $\Pi_m = D_m^\dagger D_m$ is a projection operator to $(\ker D_m)^\perp$, we have $u^m \in (\ker D_m)^\perp$ if and only if $\Pi_m u^m = u^m$, and the claim follows. ☐

Using Lemma 4.5, we next show below that for each game $\mathcal{G}$ there exists a unique strategically equivalent game which is normalized (contains no nonstrategic information).

Lemma 4.6. *Let* $\mathcal{G}$ *be a game with utilities* $\{u^m\}_{m\in\mathcal{M}}$*. Then there exists a unique game* $\hat{\mathcal{G}}$ *which (i) has the same pairwise comparison function as* $\mathcal{G}$ *and (ii) is normalized. Moreover the utilities* $\hat{u} = \{\hat{u}^m\}_{m\in\mathcal{M}}$ *of* $\hat{\mathcal{G}}$ *satisfy* $\hat{u}^m = \Pi_m u^m$ *for all* m*.*

Proof. To prove the claim, we show that given $u = \{u^m\}_{m\in\mathcal{M}}$, the game with the collection of utilities $D^\dagger D u = \Pi u$, is a normalized game with the same pairwise comparisons, and moreover there cannot be another normalized game which has the same pairwise comparisons.

Since Π is a projection operator, it follows that $\Pi\Pi u = \Pi u$, and hence, Lemma 4.5 implies that Πu is normalized. Additionally, using properties of the pseudoinverse we have $D\Pi u = DD^\dagger D u = Du$, thus Πu and u have the same pairwise comparison.

[3]We note, however, that payoff-specific information such as efficiency notions are not necessarily preserved; see Section 5.3.

Let $v \in C_0^M$ denote the collection of payoff functions of a game which is normalized and has the same pairwise comparison as u. It follows that $Dv = Du = D\Pi u$, and hence $v - \Pi u \in \ker D$. On the other hand, since both v and Πu are normalized, by Lemma 4.5, we have $v, \Pi u \in (\ker D)^{\perp}$, and thus $v - \Pi u \in (\ker D)^{\perp}$. Therefore, it follows that $v - \Pi u = 0$, hence Πu is the collection of utility functions of the unique normalized game, which has the same pairwise comparison function as $\mathcal{G}$. By Lemma 4.4, $\Pi u = \{\Pi_m u^m\}$, hence the claim follows. □

We are now ready to define the subspaces of games that will appear in our decomposition result.

Definition 4.2. *The potential subspace $\mathcal{P}$, the harmonic subspace $\mathcal{H}$ and the nonstrategic subspace $\mathcal{N}$ are defined as:*

$$\begin{aligned}
\mathcal{P} &\triangleq \left\{u \in C_0^M \mid u = \Pi u \text{ and } Du \in \text{im } \delta_0\right\} \\
\mathcal{H} &\triangleq \left\{u \in C_0^M \mid u = \Pi u \text{ and } Du \in \ker \delta_0^*\right\} \\
\mathcal{N} &\triangleq \left\{u \in C_0^M \mid u \in \ker D\right\}.
\end{aligned} \tag{28}$$

Since the operators involved in the above definitions are linear, it follows that the sets $\mathcal{P}$, $\mathcal{H}$ and $\mathcal{N}$ are indeed subspaces.

Lemma 4.5 implies that the games in $\mathcal{P}$ and $\mathcal{H}$ are normalized (contain no nonstrategic information). The flows generated by the games in these two subspaces are related to the flows induced by the Helmholtz decomposition. It follows from the definitions that the flows generated by a game in $\mathcal{P}$ are in the image space of δ_0 and the flows generated by a game in $\mathcal{H}$ are in the kernel of δ_0^*. Thus, $\mathcal{P}$ corresponds to the set of normalized games, which have globally consistent pairwise comparisons. Due to (25), the pairwise comparisons of games do not have locally inconsistent components, thus Theorem 3.1 implies that $\mathcal{H}$ corresponds to the set of normalized games, which have globally inconsistent but locally consistent pairwise comparisons. Hence, from the perspective of the Helmholtz decomposition, the flows generated by games in $\mathcal{P}$ and $\mathcal{H}$ are gradient and harmonic flows respectively. On the other hand the flows generated by games in $\mathcal{N}$ are always zero, since $Du = 0$ in such games.

As discussed in the previous section the image spaces of D_m are orthogonal. Thus, since by definition $Du = \sum_{m \in \mathcal{M}} D_m u^m$, it follows that $u = \{u^m\}_{m \in \mathcal{M}} \in \ker D$ if and only if $u^m \in \ker D_m$ for all $m \in \mathcal{M}$. Using these facts together with Lemma 4.5, we obtain the following alternative description of the subspaces of games:

$$\begin{aligned}
\mathcal{P} &= \left\{\{u^m\}_{m \in \mathcal{M}} \mid D_m u^m = D_m \phi \text{ and } \Pi_m u^m = u^m \text{ for all } m \in \mathcal{M} \text{ and some } \phi \in C_0\right\} \\
\mathcal{H} &= \left\{\{u^m\}_{m \in \mathcal{M}} \mid \delta_0^* \sum_{m \in \mathcal{M}} D_m u^m = 0 \text{ and } \Pi_m u^m = u^m \text{ for all } m \in \mathcal{M} \ \right\} \\
\mathcal{N} &= \left\{\{u^m\}_{m \in \mathcal{M}} \mid D_m u^m = 0 \text{ for all } m \in \mathcal{M} \ \right\}.
\end{aligned} \tag{29}$$

The main result of this section shows that not only these subspaces have distinct properties in terms of the flows they generate, but in fact they form a direct sum decomposition of the space of games. We exploit the Helmholtz decomposition (Theorem 3.1) for the proof.

Theorem 4.1. *The space of games $\mathcal{G}_{\mathcal{M},E}$ is a direct sum of the potential, harmonic and nonstrategic subspaces, i.e., $\mathcal{G}_{\mathcal{M},E} = \mathcal{P} \oplus \mathcal{H} \oplus \mathcal{N}$. In particular, given a game with utilities $u = \{u^m\}_{m \in \mathcal{M}}$, it can be uniquely decomposed in three components:*

- **Potential Component:** $u_P \triangleq D^{\dagger}\delta_0\delta_0^{\dagger}Du$
- **Harmonic Component:** $u_H \triangleq D^{\dagger}(I - \delta_0\delta_0^{\dagger})Du$

- **Nonstrategic Component:** $u_N \triangleq (I - D^\dagger D)u$

where $u_P + u_H + u_N = u$, *and* $u_P \in \mathcal{P}$, $u_H \in \mathcal{H}$, $u_N \in \mathcal{N}$. *The potential function associated with* u_P *is* $\phi \triangleq \delta_0^\dagger Du$.

Proof. The decomposition of $\mathcal{G}_{\mathcal{M},E}$ described above follows directly from pulling back the Helmholtz decomposition of C_1 through the map D, and removing the kernel of D; see Figure 6.

$$\begin{tikzcd}
\mathcal{G}_{\mathcal{M},E} \cong C_0^M \arrow[dr, "D"] & & \\
C_0 \arrow[r, "\delta_0"'] & C_1 \arrow[r, "\delta_1"'] & C_2
\end{tikzcd}$$

Figure 6: The Helmholz decomposition of the space of flows (C_1) can be pulled back through D to a direct sum decomposition of the space of games ($\mathcal{G}_{\mathcal{M},E}$).

The components of the decomposition clearly satisfy $u_P + u_H + u_N = u$. We verify the inclusion properties, according to (28). Both u_P and u_H are orthogonal to $\mathcal{N} = \ker D$, since they are in the range of $D^\dagger$.

- For the potential component, let $\phi \in C_0$ be such that $\phi = \delta_0^\dagger Du$. Then, we have $Du_P \in \text{im}\,(\delta_0)$, since
$$Du_P = DD^\dagger \delta_0 \delta_0^\dagger Du = \delta_0 \delta_0^\dagger Du = \delta_0 \phi,$$
where we used the definition of u_P, the property (v) in Lemma 4.4 and the definition of ϕ, respectively. This equality also implies that ϕ is the potential function associated with u_P.

- For the harmonic component u_H, we have $Du_H \in \ker \delta_0^*$:
$$\delta_0^* Du_H = \delta_0^* DD^\dagger (I - \delta_0 \delta_0^\dagger) Du = \delta_0^* (I - \delta_0 \delta_0^\dagger) Du = 0,$$
as follows from the definition of u_H, the property (v) in Lemma 4.4, and properties of the pseudoinverse.

- To check that $u_N \in \mathcal{N}$, we have
$$Du_N = D(I - D^\dagger D)u = (D - DD^\dagger D)u = 0.$$

In order to prove that the direct sum decomposition property holds, we assume that there exists $\hat{u}_P \in \mathcal{P}$, $\hat{u}_H \in \mathcal{H}$ and $\hat{u}_N \in \mathcal{N}$ such that $\hat{u}_P + \hat{u}_H + \hat{u}_N = 0$. Observe that $I - D^\dagger D$ is a projection operator to the kernel of D. Thus, from the definition of the subspaces $\mathcal{P}$, $\mathcal{H}$ and $\mathcal{N}$, it follows that $(I - D^\dagger D)\hat{u}_N = \hat{u}_N$ and $(I - D^\dagger D)\hat{u}_P = (I - D^\dagger D)\hat{u}_H = 0$. Similarly, $\delta_0 \delta_0^\dagger$ is a projection operator to the image of δ_0. Since by definition $D\hat{u}_P \in \text{im}\ \delta_0$, and $D\hat{u}_H \in \ker \delta_0^* = (\text{im}\ \delta_0)^\perp$, it follows that $\delta_0 \delta_0^\dagger D\hat{u}_P = D\hat{u}_P$ and $\delta_0 \delta_0^\dagger D\hat{u}_H = 0$.

Using these identities, it follows that
$$\begin{aligned}
(D^\dagger \delta_0 \delta_0^\dagger D)(\hat{u}_P + \hat{u}_H + \hat{u}_N) &= \hat{u}_P \\
D^\dagger (I - \delta_0 \delta_0^\dagger) D(\hat{u}_P + \hat{u}_H + \hat{u}_N) &= \hat{u}_H \\
(I - D^\dagger D)(\hat{u}_P + \hat{u}_H + \hat{u}_N) &= \hat{u}_N,
\end{aligned}$$

Since, $\hat{u}_P + \hat{u}_H + \hat{u}_N = 0$ by our assumption, it follows that $\hat{u}_P = \hat{u}_H = \hat{u}_N = 0$, and hence the direct sum decomposition property follows. □

The pseudoinverse of a linear operator L, projects its argument to the image space of L, and then, pulls the projection back to the the domain of L. Thus, intuitively, the potential function $\phi = \delta_0^\dagger Du$, defined in the theorem, is such that the gradient flow associated with it ($\delta_0\phi$) approximates the flow in the original game (Du), in the best possible way. The potential component of the game can be identified by pulling back this gradient flow through D to C_0^M. The harmonic component can similarly be obtained using the harmonic flow.

Since $\delta_0 = \sum_m D_m$, it follows that $\phi = \delta_0^\dagger Du = (\sum_m D_m)^\dagger \sum_m D_m u^m$. Thus, Lemma 4.4 (ii), and identities $\Delta_{0,m} = D_m^* D_m$ and $\Delta_0 = \sum_m \Delta_{0,m}$ imply that

$$\phi = \Delta_0^\dagger \sum_{m \in \mathcal{M}} \Delta_{0,m} u^m.$$

Additionally, from Lemma 4.4 (iii) and (iv) it follows that $D^\dagger \delta_0 = [D_1^\dagger D_1; \ldots; D_M^\dagger D_M] = [\Pi_1; \ldots; \Pi_M]$ and $D^\dagger D = \Pi = diag(\Pi_1, \ldots, \Pi_M)$. Using these identities, the utility functions of components of a game can alternatively be expressed as follows:

- **Potential Component:** $u_P^m = \Pi_m \phi$, for all $m \in \mathcal{M}$,
- **Harmonic Component:** $u_H^m = \Pi_m u^m - \Pi_m \phi$, for all $m \in \mathcal{M}$,
- **Nonstrategic Component:** $u_N^m = (I - \Pi_m) u^m$, for all $m \in \mathcal{M}$.

It can be seen that the definitions of the subspaces do not rely on the inner product in C_0^M. Thus, the direct sum property implies that the decomposition is canonical, i.e., it is independent of the inner product used in C_0^M. The above expressions provide closed form solutions for the utility functions in the decomposition, without reference to this inner product. We show in Section 6 that our decomposition is indeed orthogonal with respect to a natural inner product in C_0^M.

Note that $\Delta_0 : C_0 \to C_0$, whereas $\delta_0 : C_0 \to C_1$. Since C_1 and C_0 are associated with the edges and the nodes of the game graph respectively, in general C_1 is higher dimensional than C_0. Therefore, calculating $\Delta_0^\dagger$ is computationally more tractable than calculating $\delta_0^\dagger$. Hence, the alternative expressions for the components of a game and the potential function ϕ, have computational benefits over using the results of Theorem 4.1 directly.

We conclude this section by characterizing the dimensions of the potential, harmonic and nonstrategic subspaces.

Proposition 4.1. *The dimensions of the subspaces $\mathcal{P}$, $\mathcal{H}$ and $\mathcal{N}$ are:*

1. $\dim(\mathcal{P}) = \prod_{m \in \mathcal{M}} h_m - 1$,
2. $\dim(\mathcal{H}) = (M-1) \prod_{m \in \mathcal{M}} h_m - \sum_{m \in \mathcal{M}} \prod_{k \neq m} h_k + 1$.
3. $\dim(\mathcal{N}) = \sum_{m \in \mathcal{M}} \prod_{k \neq m} h_k$.

Proof. Lemma 4.2 provides a basis for kernel of D_m and $\dim(\ker(D_m)) = |E^{-m}|$, i.e., the cardinality of the basis is equal to $|E^{-m}|$. By definition $\mathcal{N} = \ker D = \prod_{m \in \mathcal{M}} \ker(D_m)$, hence

$$\dim(\mathcal{N}) = \sum_{m \in \mathcal{M}} \dim(\ker(D_m)) = \sum_{m \in \mathcal{M}} |E^{-m}| = \sum_{m \in \mathcal{M}} \prod_{k \neq m} h_k. \tag{30}$$

Next consider the subspace $\mathcal{P}$ of normalized potential games. By definition, the games in this set generate globally consistent flows. Moreover, by Lemma 4.6 it follows that there is a unique game in $\mathcal{P}$, which generates a given gradient flow. Thirdly, note that any globally consistent flow can be obtained as $\delta_0\phi$ for some $\phi \in C_0$, and the game $\{\Pi_m\phi\}_{m\in\mathcal{M}} \in \mathcal{P}$ generates the same flows as $\delta_0\phi$. These three facts imply that there is a linear bijective mapping between the games in $\mathcal{P}$ and the globally consistent flows, and hence the dimension of $\mathcal{P}$ is equal to the dimension of the globally consistent flows.

On the other hand, the dimension of the globally consistent flows is equivalent to $\dim(\text{im }(\delta_0))$. Since $\Delta_0 = \delta_0^*\delta_0$ it follows that $\ker(\delta_0) \subset \ker(\Delta_0)$. By Lemma 4.3 it follows that $\ker(\Delta_0) = \{f \in C_0 \mid f(\mathbf{p}) = c \in \mathbb{R}, \text{for all } \mathbf{p} \in E\ \}$. It follows from the definition of δ_0 that $\delta_0 f = 0$ for all $f \in \ker(\Delta_0)$. These facts imply that $\ker(\delta_0) = \ker(\Delta_0)$ and hence $\dim(\ker(\delta_0)) = 1$. Since δ_0 is a linear operator it follows that $\dim(\text{im }(\delta_0)) = \dim(C_0) - \dim(\ker(\delta_0)) = |E| - 1 = \prod_{m\in\mathcal{M}} h_m - 1$.

Finally observe that $\dim(\mathcal{G}_{\mathcal{M},E}) = \dim(C_0^M) = M|E| = M\prod_{m\in\mathcal{M}} h_m$. Theorem 4.1 implies that $\dim(\mathcal{G}_{\mathcal{M},E}) = \dim(\mathcal{P}) + \dim(\mathcal{H}) + \dim(\mathcal{N})$. Therefore, it follows that $\dim(\mathcal{H}) = (M-1)\prod_{m\in\mathcal{M}} h_m - \sum_{m\in\mathcal{M}}\prod_{k\neq m} h_k + 1$. □

4.3 Bimatrix Games

We conclude this section by providing an explicit decomposition result for bimatrix games, i.e., finite games with two players. Consider a bimatrix game, where the payoff matrix of the row player is given by A, and that of the column player is given by B; that is, when the row player plays i and the column player plays j, the row player's payoff is equal to A_{ij} and the column player's payoff is equal to B_{ij}.

Assume that both the row player and the column player have the same number h of strategies. It immediately follows from Proposition 4.1 that $\dim\mathcal{P} = h^2 - 1$, $\dim\mathcal{H} = (h-1)^2$ and $\dim\mathcal{N} = 2h$. For simplicity, we further assume that the payoffs are normalized[4]. Thus, the definition of normalized games implies that $\mathbf{1}^T A = B\mathbf{1} = 0$, where $\mathbf{1}$ denotes the vector of ones. Denote by A_P (B_P) and A_H (B_H) respectively, the payoff matrices of the row player (column player) in the potential and harmonic components of the game. Using our decomposition result (Theorem 4.1), it follows that

$$(A_P, B_P) = (S+\Gamma, S-\Gamma), \qquad (A_H, B_H) = (D-\Gamma, -D+\Gamma), \tag{31}$$

where $S = \frac{1}{2}(A+B)$, $D = \frac{1}{2}(A-B)$, $\Gamma = \frac{1}{2h}(A\mathbf{1}\mathbf{1}^T - \mathbf{1}\mathbf{1}^T B)$. Interestingly, the potential component of the game relates to the average of the payoffs in the original game and the harmonic component relates to the difference in payoffs of players. The Γ term ensures that the potential and harmonic components do not contain nonstrategic information. We use the above characterization in the next example for obtaining explicit payoff matrices for each of the game components.

Example 4.1 (Generalized Rock-Paper-Scissors)**.** *The payoff matrix of the generalized Rock-Paper-Scissors (RPS) game is given in Table 2a. Tables 2b, 2c and 2d include the nonstrategic, potential and the harmonic components of the game. The special case where* $x = y = z = \frac{1}{3}$ *corresponds to the celebrated RPS game. Note that in this case, the potential component of the game is equal to zero.*

[4] Lemma 4.2 and Lemma 4.6 imply that if the payoffs are not normalized, the normalized payoffs can be obtained as $(A - \frac{1}{h}\mathbf{1}\mathbf{1}^T A,\ B - \frac{1}{h}B\mathbf{1}\mathbf{1}^T)$.

	R	P	S
R	$0, 0$	$-3x, 3x$	$3y, -3y$
P	$3x, -3x$	$0, 0$	$-3z, 3z$
S	$-3y, 3y$	$3z, -3z$	$0, 0$

(a) Generalized RPS Game

	R	P	S
R	$(x-y),(x-y)$	$(z-x),(x-y)$	$(y-z),(x-y)$
P	$(x-y),(z-x)$	$(z-x),(z-x)$	$(y-z),(z-x)$
S	$(x-y),(y-z)$	$(z-x),(y-z)$	$(y-z),(y-z)$

(b) Nonstrategic Component

	R	P	S
R	$(y-x),(y-x)$	$(y-x),(x-z)$	$(y-x),(z-y)$
P	$(x-z),(y-x)$	$(x-z),(x-z)$	$(x-z),(z-y)$
S	$(z-y),(y-x)$	$(z-y),(x-z)$	$(z-y),(z-y)$

(c) Potential Component

	R	P	S
R	$0, 0$	$-(x+y+z),(x+y+z)$	$(x+y+z),-(x+y+z)$
P	$(x+y+z),-(x+y+z)$	$0, 0$	$-(x+y+z),(x+y+z)$
S	$-(x+y+z),(x+y+z)$	$(x+y+z),-(x+y+z)$	$0, 0$

(d) Harmonic Component

Table 2: Generalized RPS game and its components.

5 Properties of the Components

In this section we study the classes of games that are naturally motivated by our decomposition. In particular, we focus on two classes of games: (i) Games with no harmonic component, (ii) Games with no potential component. We show that the first class is equivalent to the well-known class of *potential games*. We refer to the games in the second class as *harmonic games*. Pictorially, we have

$$\underbrace{\mathcal{P} \quad \oplus \quad \overbrace{\mathcal{N}}^{\text{}}}_{\text{Potential games}}\!\!\!\!\!\!\overbrace{\phantom{\mathcal{N}} \quad \oplus \quad \mathcal{H}}^{\text{Harmonic games}}.$$

In Sections 5.1 and 5.2, we explain these facts, and develop and discuss several properties of these classes of games, with particular emphasis on their equilibria. Since potential games have been extensively studied in the literature, our main focus is on harmonic games. In Section 5.3, we elaborate on the effect of the nonstrategic component. Potential and harmonic games are related to other well-known classes of games, such as the zero-sum games and identical interest games. In Section 5.4, we discuss this relation, in the context of bimatrix games. As a preview, in Table 3, we summarize some of the properties of potential and harmonic games that we obtain in the subsequent sections.

5.1 Potential Games

Since the seminal paper of Monderer and Shapley [32], potential games have been an active research topic. The desirable equilibrium properties and structure of these games played a key role in this. In this section we explain the relation of the potential games to the decomposition in Section 4 and briefly discuss their properties.

Recall from Definition 2.1 that a game is a potential game if and only if there exists some $\phi \in C_0$ such that $Du = \delta_0 \phi$. This condition implies that a game is potential if and only if the associated flow is globally consistent. Thus, it can be seen from the definition of the subspaces and

	Potential Games	Harmonic Games
Subspaces	$\mathcal{P} \oplus \mathcal{N}$	$\mathcal{H} \oplus \mathcal{N}$
Flows	Globally consistent	Locally consistent but globally inconsistent
Pure NE	Always Exists	Generically does not exist
Mixed NE	Always Exists	-Uniformly mixed strategy is always a mixed NE -Players do not strictly prefer their equilibrium strategies.
Special Cases	–	-(two players) Set of mixed Nash equilibria coincides with the set of correlated equilibria -(two players & equal number of strategies) Uniformly mixed strategy is the unique mixed NE

Table 3: Properties of potential and harmonic games.

Theorem 4.1 that the set of potential games is actually equivalent to $\mathcal{P} \oplus \mathcal{N}$. For future reference, we summarize this result in the following theorem.

Theorem 5.1. *The set of potential games is equal to the subspace $\mathcal{P} \oplus \mathcal{N}$.*

Theorem 5.1 implies that potential games are games which only have potential and nonstrategic components. Since this set is a subspace, one can consider *projections* onto the set of potential games, i.e., it is possible to find the closest potential game to a given game. We pursue the idea of projection in Section 6. Using the previous theorem we next find the dimension of the subspace of potential games.

Corollary 5.1. *The subspace of potential games, $\mathcal{P} \oplus \mathcal{N}$, has dimension $\prod_{m\in\mathcal{M}} h_m + \sum_{m\in\mathcal{M}} \prod_{k\neq m} h_k - 1$.*

Proof. The result immediately follows from Theorem 5.1 and Proposition 4.1. $\square$

We next provide a brief discussion of the equilibrium properties of potential games.

Theorem 5.2 ([32])**.** *Let $\mathcal{G} = \langle \mathcal{M}, \{E^m\}, \{u^m\} \rangle$ be a potential game and ϕ be a corresponding potential function.*

1. *The equilibrium set of $\mathcal{G}$ coincides with the equilibrium set of $\mathcal{G}_\phi \triangleq \langle \mathcal{M}, \{E^m\}, \{\phi\} \rangle$.*
2. *$\mathcal{G}$ has a pure Nash equilibrium.*

The first result follows from the fact that the games $\mathcal{G}$ and $\mathcal{G}_\phi$ are strategically equivalent. Alternatively, the preferences in $\mathcal{G}$ are aligned with the global objective denoted by the potential function ϕ. The second result is implied by the first one since in finite games the potential function ϕ necessarily has a maximum, and the maximum is a Nash equilibrium of $\mathcal{G}_\phi$. These results indicate that potential games can be analyzed by an equivalent game where each player has the same utility function ϕ. The second game is easy to analyze since when agents have the same objective, the game is similar to an optimization problem with objective function ϕ.

Another desirable property of potential games relates to their dynamical properties. An important question in game theory is how a game reaches an equilibrium. This question is usually answered by theoretical models of player dynamics. For general games, “natural” player dynamics do not necessarily converge to an equilibrium and various counterexamples are provided in the literature [10, 22]. However, it is known that some of the well-known dynamics such as fictitious play, best-response dynamics (and their variants) converges in potential games [32, 27, 47, 4, 28, 10, 19, 39]. The results for convergence in potential games can be extended to "near-potential" games using our decomposition framework and these results are discussed in [3].

5.2 Harmonic Games

In this section, we focus on games in which the potential component is zero, hence the strategic interactions are governed only by the harmonic component. We refer to such games as *harmonic games*, i.e., a game $\mathcal{G}$ is a harmonic game if $\mathcal{G} \in \mathcal{H} \oplus \mathcal{N}$.

This section studies the properties of equilibria of harmonic games. We first characterize the Nash equilibria of such games, and show that generically they do not have a pure Nash equilibrium. We further consider mixed Nash and correlated equilibria, and show how the properties of harmonic games restrict the possible set of equilibria.

5.2.1 Pure Equilibria

In this section, we focus on pure Nash equilibria in harmonic games. Additionally, we characterize the dimension of the space of harmonic games, $\mathcal{H} \oplus \mathcal{N}$.

We first show that at a pure Nash equilibrium of a harmonic game, all players are indifferent between *all* of their strategies.

Lemma 5.1. *Let $\mathcal{G} = \langle \mathcal{M}, \{E^m\}, \{u^m\} \rangle$ be a harmonic game and $\mathbf{p}$ be a pure Nash equilibrium. Then,*

$$u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = u^m(\mathbf{q}^m, \mathbf{p}^{-m}) \quad \text{for all } m \in \mathcal{M} \text{ and } \mathbf{q}^m \in E^m. \tag{32}$$

Proof. By definition, in harmonic games the utility functions $u = \{u^m\}$ satisfy the condition $\delta_0^* D u = 0$. By (12) and (20), $\delta_0^* D u$ evaluated at $\mathbf{p}$ can be expressed as,

$$\sum_{m \in \mathcal{M}} \sum_{\mathbf{q} | (\mathbf{p}, \mathbf{q}) \in A^m} (u^m(\mathbf{p}) - u^m(\mathbf{q})) = 0. \tag{33}$$

Since $\mathbf{p}$ is a Nash equilibrium it follows that $u^m(\mathbf{p}) - u^m(\mathbf{q}) \geq 0$ for all $(\mathbf{p}, \mathbf{q}) \in A^m$ and $m \in \mathcal{M}$. Combining this with (33) it follows that $u^m(\mathbf{p}) - u^m(\mathbf{q}) = 0$ for all $(\mathbf{p}, \mathbf{q}) \in A^m$ and $m \in \mathcal{M}$. Observing that $(\mathbf{p}, \mathbf{q}) \in A^m$ if and only if $\mathbf{q} = (\mathbf{q}^m, \mathbf{p}^{-m})$, the result follows. □

Using this result we next prove that harmonic games generically do not have pure Nash equilibria. By "generically", we mean that it is true for almost all harmonic games, except possibly for a set of measure zero (for instance, the trivial game where all utilities are zero is harmonic, and clearly has pure Nash equilibria).

Proposition 5.1. *Harmonic games generically do not have pure Nash equilibria.*

Proof. Define $\mathcal{G}_\mathbf{p} \subset \mathcal{H} \oplus \mathcal{N}$ as the set of harmonic games for which $\mathbf{p}$ is a pure Nash equilibrium. Observe that $\cup_{\mathbf{p} \in E} \mathcal{G}_\mathbf{p}$ is the set of all harmonic games which have a pure Nash equilibria. We show that $\mathcal{G}_\mathbf{p}$ is a lower dimensional subspace of the space of harmonic games for each $\mathbf{p} \in E$. Since the set of harmonic games with pure Nash equilibrium is a finite union of lower dimensional subspaces it follows that generically harmonic games do not have pure Nash equilibria.

By Lemma 5.1 it follows that

$$\mathcal{G}_\mathbf{p} = (\mathcal{H} \oplus \mathcal{N}) \cap \{\{u^m\}_{m \in \mathcal{M}} | u^m(\mathbf{p}) = u^m(\mathbf{q}), \text{ for all } \mathbf{q} \text{ such that } (\mathbf{p}, \mathbf{q}) \in A^m \text{ and } m \in \mathcal{M} \ \}.$$

Hence $\mathcal{G}_{\mathbf{p}}$ is a subspace contained in $\mathcal{H} \oplus \mathcal{N}$. It immediately follows that $\mathcal{G}_{\mathbf{p}}$ is a lower dimensional subspace if we can show that there exists harmonic games which are not in $\mathcal{G}_{\mathbf{p}}$, i.e., in which $\mathbf{p}$ is not a pure Nash equilibrium.

Assume that $\mathbf{p}$ is a pure Nash equilibrium in all harmonic games. Since $\mathbf{p}$ is arbitrary this holds only if all strategy profiles are pure Nash equilibria in harmonic games. If all strategy profiles are Nash equilibria, by Lemma 5.1 it follows that the pairwise ranking function is equal to zero in harmonic games, hence $\mathcal{H} \oplus \mathcal{N} \subset \mathcal{N}$. We reach a contradiction since dimension of $\mathcal{H}$ is larger than zero.

Therefore, $\mathcal{G}_{\mathbf{p}}$ is a strict subspace of the space of harmonic games, and thus harmonic games generically do not have pure Nash equilibria. □

We conclude this section by a dimension result that is analogous to the result obtained for potential games.

Theorem 5.3. *The set of harmonic games, $\mathcal{H} \oplus \mathcal{N}$, has dimension $(M-1)\prod_{m \in \mathcal{M}} h_m + 1$.*

Proof. The result immediately follows from Theorem 4.1 and Proposition 4.1. □

5.2.2 Mixed Nash and Correlated Equilibria in Harmonic Games

In the previous section we showed that harmonic games generically do not have pure Nash equilibria. In this section, we study their mixed Nash and correlated equilibria. In particular, we show that in harmonic games, the mixed strategy profile, in which players uniformly randomize over their strategies is always a mixed Nash equilibrium. Additionally, in the case of two-player harmonic games mixed Nash and correlated equilibria coincide, and if players have equal number of strategies the uniformly mixed strategy profile is the unique correlated equilibrium of the game. Before we discuss the details of these results, we next provide some preliminaries and notation.

We denote the set of probability distributions on E by ΔE. Given $x \in \Delta E$, $x(\mathbf{p})$ denotes the probability assigned to $\mathbf{p} \in E$. Observe that for all $x \in \Delta E$, $\sum_{\mathbf{p} \in E} x(\mathbf{p}) = 1$, and $x(\mathbf{p}) \geq 0$. Similarly for each player $m \in \mathcal{M}$, ΔE^m denotes the set of probability distributions on E^m and for $x^m \in \Delta E^m$, $x^m(\mathbf{p}^m)$ is the probability assigned to strategy $\mathbf{p}^m \in E^m$. As before all $x^m \in \Delta E^m$ satisfies $\sum_{\mathbf{p}^m \in E^m} x^m(\mathbf{p}^m) = 1$ and $x^m(\mathbf{p}^m) \geq 0$. We refer to the distribution $x^m \in \Delta E^m$ as a mixed strategy of player $m \in \mathcal{M}$ and the collection $x = \{x^m\}_m$ as a mixed strategy profile. Note that $\{x^m\}_m \in \prod_m \Delta E^m \subset \Delta E$. Mixed strategies of all players but the mth one is denoted by x^{-m}.

With some abuse of the notation, we define the mixed extensions of the utility functions $u^m : \prod_m \Delta E^m \to \mathbb{R}$ such that for any $x \in \prod_m \Delta E^m$,

$$u^m(x) = \sum_{\mathbf{p} \in E} u^m(\mathbf{p}) \prod_{k \in \mathcal{M}} x^k(\mathbf{p}^k). \tag{34}$$

Similarly, if player m uses pure strategy $\mathbf{q}^m$ and the other players use the mixed strategies x^{-m} we denote the payoff of player m by,

$$u^m(\mathbf{q}^m, x^{-m}) = \sum_{\mathbf{p}^{-m} \in E^{-m}} u^m(\mathbf{q}^m, \mathbf{p}^{-m}) \prod_{k \in \mathcal{M}, k \neq m} x^k(\mathbf{p}^k). \tag{35}$$

Using this notation we can define the solution concepts.

Definition 5.1 (Mixed Nash / Correlated Equilibrium)**.** *Consider the game $\langle \mathcal{M}, \{E^m\}, \{u^m\} \rangle$.*

1. *A mixed strategy profile $x = \{x^m\}_m \in \prod_m \Delta E^m$ is a* mixed Nash equilibrium *if for all $m \in \mathcal{M}$ and $\mathbf{p}^m \in E^m$, $u^m(x^m, x^{-m}) \geq u^m(\mathbf{p}^m, x^{-m})$.*

2. A probability distribution $x \in \Delta E$ is a correlated equilibrium *if for all $m \in \mathcal{M}$ and $\mathbf{p}^m, \mathbf{q}^m \in E^m$, $\sum_{\mathbf{p}^{-m}} (u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m}))\, x(\mathbf{p}^m, \mathbf{p}^{-m}) \geq 0$.*

From these definitions it can be seen that every mixed Nash equilibrium is a correlated equilibrium where the corresponding distribution $x \in \prod_m \Delta E^m \subset \Delta E$ is a product distribution, i.e., it satisfies $x(\mathbf{p}) = \prod_m x^m(\mathbf{p}^m)$

These definitions also imply that similar to Nash equilibrium, the conditions for mixed Nash and correlated equilibria can be expressed only in terms of pairwise comparisons. Therefore, these equilibrium sets are independent of the nonstrategic components of games.

We next obtain an alternative characterization of correlated equilibria in normalized harmonic games. This characterization will be more convenient when studying the equilibrium properties of harmonic games, as it is expressed in terms of equalities, instead of inequalities.

Proposition 5.2. *Consider a normalized harmonic game, $\mathcal{G} = \langle \mathcal{M}, \{u^m\}, E^m \rangle$ and a probability distribution $x \in \Delta E$. The following are equivalent:*

(i) x is a correlated equilibrium.

(ii) For all $\mathbf{p}^m$, $\mathbf{q}^m$ and $m \in \mathcal{M}$,

$$\sum_{\mathbf{p}^{-m}} \left(u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m})\right) x(\mathbf{p}^m, \mathbf{p}^{-m}) = 0. \tag{36}$$

(iii) For all $\mathbf{p}^m$, $\mathbf{q}^m$ and $m \in \mathcal{M}$,

$$\sum_{\mathbf{p}^{-m}} u^m(\mathbf{q}^m, \mathbf{p}^{-m}) x(\mathbf{p}^m, \mathbf{p}^{-m}) = 0. \tag{37}$$

Proof. We prove the claim, by first showing (i) and (ii) are equivalent and then establishing the equivalence (ii) and (iii).

By the definition of correlated equilibrium, (36) implies that x is a correlated equilibrium. To see that any correlated equilibrium of $\mathcal{G}$ satisfies (36), assume $x \in \Delta E$ is a correlated equilibrium. Since the game is a harmonic game, by definition, the utility functions $u = \{u^m\}$ satisfy the condition $\delta_0^* D u = 0$. Using (12) and (20), this condition can equivalently be expressed as

$$\sum_{m \in \mathcal{M}} \sum_{\mathbf{q}^m \in E^m} u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = 0 \quad \text{for all } \mathbf{p} \in E. \tag{38}$$

Thus, it follows that

$$\begin{aligned} 0 &= \sum_{\mathbf{p} \in E} x(\mathbf{p}) \sum_{m \in \mathcal{M}} \sum_{\mathbf{q}^m \in E^m} u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) \\ &= \sum_{m \in \mathcal{M}} \sum_{\mathbf{q}^m \in E^m} \sum_{\mathbf{p}^m \in E^m} \sum_{\mathbf{p}^{-m} \in E^{-m}} x(\mathbf{p}^m, \mathbf{p}^{-m}) \left(u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m})\right). \end{aligned} \tag{39}$$

Since x is a correlated equilibrium, $\sum_{\mathbf{p}^{-m} \in E^{-m}} x(\mathbf{p}^m, \mathbf{p}^{-m}) \left(u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m})\right) \leq 0$ for all $\mathbf{p}^m$, $\mathbf{q}^m$ and $m \in \mathcal{M}$. Hence, (39) implies that

$$\sum_{\mathbf{p}^{-m} \in E^{-m}} x(\mathbf{p}^m, \mathbf{p}^{-m}) \left(u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m})\right) = 0$$

for all $\mathbf{p}^m$, $\mathbf{q}^m$ and $m \in \mathcal{M}$. Thus, we conclude (i) and (ii) are equivalent.

To see the equivalence of (ii) and (iii), observe that (iii) immediately implies (ii). Assume (ii) holds, then writing (36) for two strategies $\mathbf{r}^m, \mathbf{q}^m \in E^m$, and subtracting these equations from each other, it follows that

$$\sum_{\mathbf{p}^{-m}} \left(u^m(\mathbf{r}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m})\right) x(\mathbf{p}^m, \mathbf{p}^{-m}) = 0. \tag{40}$$

Since $\mathbf{r}^m$ and $\mathbf{q}^m$ are arbitrary it follows that for all $\mathbf{q}^m \in E^m$

$$\sum_{\mathbf{p}^{-m}} u^m(\mathbf{q}^m, \mathbf{p}^{-m}) x(\mathbf{p}^m, \mathbf{p}^{-m}) = c_{\mathbf{p}^m}, \tag{41}$$

for some $c_{\mathbf{p}^m} \in \mathbb{R}$. Since the game is normalized, we have $\sum_{\mathbf{q}^m} u^m(\mathbf{q}^m, \mathbf{p}^{-m}) = 0$. Thus summing (41) over $\mathbf{q}^m$ it follows that $c_{\mathbf{p}^m} = 0$, and hence (ii) implies (iii).

Therefore we conclude that (i), (ii) and (iii) are equivalent for normalized harmonic games. $\square$

Note that in the above proof, we used the assumption that the game is normalized, only when establishing the equivalence of (ii) and (iii). Therefore, it can be seen that (i) and (ii) are equivalent for all harmonic games.

The above proposition implies that the correlated equilibria of harmonic games correspond to the intersection of the probability simplex with a subspace defined by the utilities in the game. Using this result, we obtain the following characterization of mixed Nash equilibria of harmonic games.

Corollary 5.2. *Let $\mathcal{G} = \langle \mathcal{M}, \{u^m\}, \{E^m\} \rangle$ be a harmonic game. The mixed strategy profile $x \in \prod_m \Delta E^m$ is a mixed Nash equilibrium if and only if,*

$$u^m(x^m, x^{-m}) = u^m(\mathbf{p}^m, x^{-m}) \qquad \textit{for all } \mathbf{p}^m \in E^m \textit{ and } m \in \mathcal{M}. \tag{42}$$

Proof. Assume that (42) holds, then clearly all players are indifferent between all their mixed strategies, hence it follows that x is a mixed Nash equilibrium of the game.

Let x be a mixed Nash equilibrium. Since each mixed Nash equilibrium is also a correlated equilibrium, from equivalence of (i) and (ii) of Proposition 5.2 for all harmonic games, it follows that for all $\mathbf{p}^m$, $\mathbf{q}^m$ and $m \in \mathcal{M}$,

$$\begin{aligned} 0 &= \sum_{\mathbf{p}^{-m}} \left(u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m})\right) x(\mathbf{p}^m, \mathbf{p}^{-m}) \\ &= x^m(\mathbf{p}^m) \sum_{\mathbf{p}^{-m}} \left(u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m})\right) \prod_{k \neq m} x^k(\mathbf{p}^k) \\ &= x^m(\mathbf{p}^m) \left(u^m(\mathbf{p}^m, x^{-m}) - u^m(\mathbf{q}^m, x^{-m})\right). \end{aligned} \tag{43}$$

Since by definition of probability distributions, there exists $\mathbf{p}^m$ such that $x^m(\mathbf{p}^m) > 0$ it follows that $u^m(\mathbf{p}^m, x^{-m}) = u^m(\mathbf{q}^m, x^{-m})$ for all $\mathbf{q}^m \in E^m$. Thus, $u^m(x^m, x^{-m}) = u^m(\mathbf{q}^m, x^{-m})$ for all $\mathbf{q}^m \in E^m$. Since m is arbitrary, the claim follows. $\square$

It is well-known that in mixed Nash equilibria of games, players are indifferent between all the pure strategies in the support of their mixed strategy (see [11]), i.e., if $x \in \prod_m \Delta E^m$ is a mixed Nash equilibrium then

$$u^m(x^m, x^{-m}) \begin{cases} = u^m(\mathbf{p}^m, x^{-m}) & \text{for all } \mathbf{p}^m \text{ such that } x^m(\mathbf{p}^m) \geq 0 \\ \geq u^m(\mathbf{p}^m, x^{-m}) & \text{for all } \mathbf{p}^m \text{ such that } x^m(\mathbf{p}^m) = 0. \end{cases} \tag{44}$$

The above corollary implies that at a mixed equilibrium of a harmonic game, each player is indifferent between all its pure strategies, including those which are not in the support of its mixed strategy.

We next define a particular mixed strategy profile, and show that it is an equilibrium in all harmonic games.

Definition 5.2 (Uniformly Mixed Strategy Profile)**.** *The* uniformly mixed strategy *of player m is a mixed strategy where player m uses $x_{\mathbf{q}^m} = \frac{1}{h_m}$ for all $\mathbf{q}^m \in E^m$. Respectively, we define the* uniformly mixed strategy profile *as the one in which all players use uniformly mixed strategies.*

Recall that rock-paper-scissors and matching pennies are examples of harmonic games, in which the uniformly mixed strategy profile is a mixed Nash equilibrium. The next theorem shows that this is a general property of harmonic games and the uniformly mixed strategy profile is always a Nash equilibrium.

Theorem 5.4. *In harmonic games, the uniformly mixed strategy profile is always a Nash equilibrium.*

Proof. Let $\mathcal{G} = \langle \mathcal{M}, \{u^m\}, \{E^m\} \rangle$ be a harmonic game, and x be the uniformly mixed strategy profile. In order to prove the claim, we first state the following useful identity (see Appendix for a proof), on the utility functions of harmonic games.

Lemma 5.2. *Let $\mathcal{G} = \langle \mathcal{M}, \{u^m\}, \{E^m\} \rangle$ be a harmonic game. Then for all $\mathbf{q}^m, \mathbf{r}^m \in E^m$, $m \in \mathcal{M}$, $\sum_{\mathbf{p}^{-m} \in E^{-m}} u^m(\mathbf{r}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m}) = 0$.*

Using this lemma, and the definition of the uniformly mixed strategy, it follows that

$$\begin{aligned} u^m(\mathbf{q}^m, x^{-m}) - u^m(\mathbf{p}^m, x^{-m}) &= \sum_{\mathbf{p}^{-m}} c_m \left(u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) \right) \\ &= c_m \sum_{\mathbf{p}^{-m}} \left(u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) \right) \\ &= 0, \end{aligned} \tag{45}$$

where $c_m = \prod_{k \neq m} x^k(\mathbf{p}^k) = \prod_{k \neq m} \frac{1}{h_k}$. Since $\mathbf{p}^m$ and $\mathbf{q}^m$ are arbitrary, (45) implies that

$$u^m(x^m, x^{-m}) = u^m(\mathbf{p}^m, x^{-m}) \tag{46}$$

for all $\mathbf{p}^m \in E^m$, and by Corollary 5.2, x is a mixed strategy Nash equilibrium. $\square$

In the sequel, we identify a basis for two-player normalized harmonic games, and through a simple dimension argument, show that this Nash equilibrium is not unique, for general harmonic games. In order to simplify the derivation of the basis result, we first provide a simple characterization of normalized harmonic games, in terms of the utility functions in the game.

Theorem 5.5. *The game $\mathcal{G}$ with utilities $u = \{u^m\}_{m \in \mathcal{M}}$ is a normalized harmonic game, i.e., it belongs to $\mathcal{H}$ if and only if $\sum_{m \in \mathcal{M}} h_m u^m = 0$ and $\Pi_m u^m = u^m$ for all $\mathbf{m} \in \mathcal{M}$, where $h_m = |E^m|$.*

Proof. By Definition 4.2, $\mathcal{G} \in \mathcal{H}$ if and only if $\Pi u = u$ and $\delta_0^* D u = 0$. Using the definitions of the operators, these conditions can alternatively be expressed as $\Pi_m u^m = u^m$ and $\delta_0^* \sum_{m \in \mathcal{M}} D_m u^m = 0$. By (23) and the orthogonality of image spaces of operators D_m, the latter equality implies that

$\sum_{m\in\mathcal{M}} D_m^* D_m u^m = \sum_{m\in\mathcal{M}} \Delta_{0,m} u^m = 0$. Using Lemma 4.1, $\Delta_{0,m} = h_m \Pi_m$, and hence it follows that $\mathcal{G} \in \mathcal{H}$, if and only if

$$\sum_{m\in\mathcal{M}} h_m \Pi_m u^m = 0 \qquad \text{and,} \qquad \Pi_m u^m = u^m \text{ for all } m. \tag{47}$$

The claim follows by replacing $\Pi_m u^m$ in the summation with u^m. □

The above theorem implies that normalized harmonic games, where players have equal number of strategies, are zero-sum games, i.e., in such games the payoffs of players add up to zero at all strategy profiles. We explore the further relations between zero-sum games and harmonic games in Section 5.4.

In the following theorem, we present a basis for two-player normalized harmonic games. The idea behind our construction is to obtain a collection of games, in which both players have "effectively" two strategies (the payoffs are equal to zero, if other strategies are played), and ensure that they are linearly independent normalized harmonic games.

Theorem 5.6. *Consider the set of two-player games where the first player has h_1 strategies and the second player has h_2 strategies. For any $i \in \{1, \ldots, h_1 - 1\}$ and $j \in \{1, \ldots, h_2 - 1\}$, define bimatrix games $\mathcal{G}^{ij}$, with payoff matrices $(h_2 A^{ij}, -h_1 A^{ij})$, where $A^{ij} \in \mathbb{R}^{h_1 \times h_2}$ is such that*

$$A^{ij}_{kl} = \begin{cases} 1 & \text{if } (k,l) = (i,j) \text{ or } (k,l) = (i+1, j+1), \\ -1 & \text{if } (k,l) = (i+1, j) = (k,l) \text{ or } (k,l) = (i, j+1), \\ 0 & \text{otherwise.} \end{cases} \tag{48}$$

The collection $\{\mathcal{G}^{ij}\}$ provides a basis of $\mathcal{H}$.

Proof. It can be seen that each $\mathcal{G}^{ij}$ is normalized, since row and column sums of A^{ij} is equal to zero. By Theorem 5.5 and (48), it also follows that $\mathcal{G}^{ij}$ belongs to $\mathcal{H}$. It can be seen from Proposition 4.1 that $\dim \mathcal{H} = (h_1 - 1)(h_2 - 1)$, is equal to the cardinality of the collection $\{\mathcal{G}^{ij}\}$. Thus, in order to prove the claim, it is sufficient to prove that

$$\sum_{i\in\{1,\ldots,h_1-1\}} \sum_{j\in\{1,\ldots,h_2-1\}} \alpha_{ij} A^{ij} = 0, \tag{49}$$

only if $\alpha_{ij} = 0$ for all i, j.

Note that A^{11} is the only matrix which has a nonzero entry in the first column and the first row. Thus, (49) implies that $\alpha_{11} = 0$. Similarly it can be seen that A^{11} and A^{12} are the only matrices which have nonzero entries in the first row and the second column, thus $\alpha_{12} = 0$. Proceeding iteratively it follows that if (49) holds, then $\alpha_{ij} = 0$ for all i, j and the claim follows. □

The next example uses the basis introduced above, to show that in harmonic games, the uniformly mixed strategy profile is not necessarily the unique mixed Nash equilibrium.

Example 5.1. *In this example we consider two-player harmonic games, where $E^1 = \{x, y\}$ and $E^2 = \{a, b, c\}$. Using Theorem 5.6, a basis for normalized two-player harmonic games is given in Tables 4a and 4b. Thus, any harmonic game with these strategy sets, can be expressed as in Table 4c. Consider some fixed α and β. As can be seen from Definition 5.1, the mixed equilibria for this game are given by*

$$(\tfrac{1}{2}, \tfrac{1}{2}) \times (\theta_1, \theta_2, \theta_3)$$

where θ_1, θ_2 and θ_3 are scalars that satisfy $\theta_1+\theta_2+\theta_3=1$, $\theta_1,\theta_2,\theta_3\geq 0$ and $\theta_1(6\alpha)+\theta_2(-6\alpha+6\beta)+\theta_3(-6\beta)=0$. Note that since there are two linear equations in three variables, this system has a continuum of solutions. Moreover, since $(\theta_1,\theta_2,\theta_3)=(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ is a solution, it follows that there is a continuum of solutions for which $\theta_1,\theta_2,\theta_3\geq 0$.

Since this is true for any α, β, we conclude that all games in $\mathcal{H}$ have uncountably many mixed equilibria. Additionally, since the nonstrategic component does not affect the equilibrium properties of a game it follows that all harmonic games on $E^1\times E^2$ (all games in $\mathcal{H}\oplus\mathcal{N}$) have uncountably many mixed Nash equilibria.

	a	b	c
x	3, -2	-3, 2	0, 0
y	-3, 2	3, -2	0, 0

(a) Basis element 1

	a	b	c
x	0, 0	3, -2	-3, 2
y	0, 0	-3, 2	3, -2

(b) Basis element 2

	a	b	c
x	3α , -2α	$-3\alpha+3\beta$, $2\alpha-2\beta$	-3β, 2β
y	-3α, 2α	$3\alpha-3\beta$, $-2\alpha+2\beta$	3β, -2β

(c) A game in $\mathcal{H}$

Table 4: Basis of $\mathcal{H}$

Using this basis, we characterize in the following theorem, the correlated equilibria in two-player harmonic games. Interestingly, our results suggest that in two-player harmonic games, the set of mixed Nash equilibria and correlated equilibria generically coincide.

Theorem 5.7. *Consider the set of two-player harmonic games where the first player has h_1 strategies and the second player has h_2 strategies. Without loss of generality assume that $h_1\geq h_2$. Generically,*

(i) Every correlated equilibrium is a mixed Nash equilibrium, where the player with minimum number of strategies uses the uniformly mixed strategy.

(ii) The dimension of the set of correlated equilibria is h_1-h_2

Proof. As discussed earlier, nonstrategic components of games do not affect the equilibrium sets. Thus, to prove that (i) and (ii) are generically true for harmonic games, it is sufficient to prove that they generically hold for normalized harmonic games.

Consider a two-player normalized harmonic game with payoff matrices (A,B), where $A,B\in\mathbb{R}^{h_1\times h_2}$. By Theorem 5.5, it follows that $A=-\frac{h_2}{h_1}B$. Denote by e_1 (similarly e_2), the h_1 (similarly h_2) dimensional vector, all entries of which are identically equal to 1. Since the game is normalized, it follows that $e_1^TA=0$ and $Be_2=-\frac{h_1}{h_2}Ae_2=0$.

Let x be a correlated equilibrium of this game. For each $\mathbf{p}^1\in E^1$, denote by $x(\mathbf{p}^1,\ \cdot)\in\mathbb{R}^{h_2}$ the vector of probabilities $[x(\mathbf{p}^1,\mathbf{p}^2)]_{\mathbf{p}^2}$. By Proposition 5.2 (iii), it follows that these vectors satisfy the condition

$$Ax(\mathbf{p}^1,\cdot)=0. \tag{50}$$

Note that we need to characterize the kernel of the payoff matrix A, to identify the correlated equilibria. For that reason, we state the following technical lemma:

Lemma 5.3. *Consider the set of normalized harmonic games in Theorem 5.7. Generically, the payoff matrices of players have their row and column ranks equal to $h_2 - 1$.*

Proof. The payoff matrices of the players satisfy $A = -\frac{h_2}{h_1}B$, so they have the same row and column rank. It follows from Theorem 5.6 that the collection of matrices $\{A^{ij}\}$ span the payoff matrices of harmonic games. It can be seen that the matrices in the span of this collection generically have row and column rank equal to $h_2 - 1$, and the claim follows. □

Using this lemma, it follows that generically the kernel of A is 1 dimensional. As shown earlier, e_2 is in the kernel of A, thus, (50), implies that generically $x(\mathbf{p}^1,\ \cdot)$ has the form $x(\mathbf{p}^1,\ \cdot) = c_{\mathbf{p}^1} e_2$, for some $c_{\mathbf{p}^1} \in \mathbb{R}$. Since x is a probability distribution, the definition of $x(\mathbf{p}^1,\ \cdot)$ implies that $x(\mathbf{p}^1, \mathbf{p}^2) = c_{\mathbf{p}^1} \geq 0$, and $\sum_{\mathbf{p}^1,\mathbf{p}^2} x(\mathbf{p}^1, \mathbf{p}^2) = h_2 \sum_{\mathbf{p}^1} c_{\mathbf{p}^1} = 1$. Thus, it follows that $x(\mathbf{p}^1, \mathbf{p}^2) = c_{\mathbf{p}^1} = \frac{\alpha_{\mathbf{p}^1}}{h_2}$, for some $\alpha_{\mathbf{p}^1} \geq 0$ such that $\sum_{\mathbf{p}^1} \alpha_{\mathbf{p}^1} = 1$. It can be seen from this description that generically, the correlated equilibria are mixed equilibria where the first player uses the probability distribution $x^1 = \alpha \triangleq [\alpha_{\mathbf{p}^1}]_{\mathbf{p}^1} \in \Delta E^1$ and the second player uses the distribution $x^2 = \left[\frac{1}{h_2}\right]_{\mathbf{p}^2}$.

Since the correlated equilibria have this form, it can be seen using Proposition 5.2 (iii) for the second player that

$$\sum_{\mathbf{p}^1} u^2(\mathbf{p}^1, \mathbf{q}^2) x(\mathbf{p}^1, \mathbf{p}^2) = \frac{1}{h_2} \sum_{\mathbf{p}^1} u^2(\mathbf{p}^1, \mathbf{q}^2) \alpha_{\mathbf{p}_1} = 0, \tag{51}$$

where $\alpha \in \Delta E^1$. The above condition can be restated using the payoff matrices as follows:

$$\alpha^T B = -\frac{h_1}{h_2} \alpha^T A = 0, \tag{52}$$

where $\alpha \in \Delta E^1$. Since, the row rank of A is $h_2 - 1$, the dimension of α that satisfies (52) is $h_1 - h_2 + 1$. Note that since α is a probability distribution, it also satisfies the condition $\alpha^T e_1 = 1$. Note that since $e_1^T A = 0$, this condition is orthogonal to the ones in (52). Hence, it follows that the dimension of α which satisfies the correlated equilibrium conditions in (52) (other than the positivity) is $h_1 - h_2$. On the other hand, $\alpha = \frac{1}{h_1} e_1$ gives a correlated equilibrium (by Theorem 5.4), thus the positivity condition does not change the dimension of the set of correlated equilibria, and the dimension is generically $h_1 - h_2$. □

An immediate implication of this theorem is the following:

Corollary 5.3. *In two-player harmonic games where players have equal number of strategies, the uniformly mixed strategy is generically the unique correlated equilibrium.*

Note that Theorem 5.7 implies that in two-player harmonic games, generically there are no correlated equilibria that are not mixed equilibria. This statement fails, when the number of players is more than two, as shown in the following theorem.

Theorem 5.8. *Consider a M-player harmonic game, where $M > 2$, and in which each player has h strategies such that $h^M > M(h^2 - 1) + 1$. The set of correlated equilibria is strictly larger than the set of mixed Nash equilibria: The set of correlated equilibria has dimension at least $h^M - 1 - Mh(h-1)$, and the set of mixed equilibria has dimension at most $M(h-1)$.*

Proof. Since each player has h strategies, the set of mixed strategies has dimension $M(h-1)$, and this is a trivial upper bound on the dimension of the set of mixed equilibria. The set of correlated equilibria, on the other hand, is defined by the equalities in Proposition 5.2. Note that there are

$Mh(h-1)$ such equalities and the dimension of ΔE is $h^M - 1$, hence the dimension of the correlated equilibria is at least $h^M - 1 - Mh(h-1)$ (by ignoring possible dependence of the equalities).

The difference in the dimensions implies that the set of correlated equilibria is strictly larger than the set of mixed equilibria. $\square$

Note that this theorem can be easily generalized to the case when players have different number of strategies. An interesting problem is to find the exact dimensions of the set of mixed Nash and correlated equilibria when there are more than two players. However, due to complicated dependence relations of the correlated equilibrium conditions in Proposition 5.2, we do not pursue this question in this paper, and leave it as a future problem.

5.3 Nonstrategic Component and Efficiency in Games

We first consider games for which the potential and harmonic components are equal to zero. In such games all pairwise comparisons are equal to zero, hence each player is indifferent between any of his strategies given any strategies of other players. It is thus immediate that all strategy profiles are Nash equilibria in such games.

More generally, from the definition of the nonstrategic component it can be seen that in any game, the pairwise comparisons are functions of only the potential and harmonic components of the game. Thus, the nonstrategic component has *no effect whatsoever* on the equilibrium properties of games. However, the nonstrategic component is of interest mainly through its effect on the efficiency properties of games, as discussed in the rest of this section. The efficiency measure we focus on is Pareto optimality.

Definition 5.3 (Pareto Optimality)**.** *A strategy profile* $\mathbf{p}$ *is Pareto optimal if and only if there does not exist another strategy profile* $\mathbf{q}$ *such that all players weakly increase their payoffs and one player strictly increases its payoff, i.e,*

$$\begin{aligned} u^m(\mathbf{q}) &\geq u^m(\mathbf{p}), && \text{for all } m \in \mathcal{M} \\ u^k(\mathbf{q}) &> u^k(\mathbf{p}), && \text{for some } k \in \mathcal{M}. \end{aligned} \tag{53}$$

We first state a preliminary lemma, which will be useful in the subsequent analysis.

Lemma 5.4. *Let* $\mathcal{G}$ *be a game with utilities* $\{u^m\}$*. There exists a game* $\hat{\mathcal{G}}$ *with utilities* $\{\hat{u}^m\}$ *such that (i) the potential and harmonic components of* $\hat{\mathcal{G}}$ *are identical to these of* $\mathcal{G}$ *and (ii) in* $\hat{\mathcal{G}}$ *all players get zero payoff at all strategy profiles that are pure Nash equilibria of* $\mathcal{G}$*.*

Proof. Let $N_{\mathcal{G}}$ be the set of pure Nash equilibria of $\mathcal{G}$. First observe that if there are m-comparable equilibria in $\mathcal{G}$ player m receives the same payoff in these equilibria, i.e., if $\mathbf{p},\mathbf{q} \in N_{\mathcal{G}}$ and $\mathbf{p} = (\mathbf{p}^m, \mathbf{p}^{-m})$, $\mathbf{q} = (\mathbf{q}^m, \mathbf{p}^{-m})$ for some m, then $u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = u^m(\mathbf{q}^m, \mathbf{p}^{-m})$. This equality holds since otherwise, player m would have incentive to improve its payoff at $\mathbf{p}$ or $\mathbf{q}$ by switching to a strategy profile with better payoff, and this contradicts with $\mathbf{p}$ and $\mathbf{q}$ being Nash equilibria of $\mathcal{G}$.

Define the game $\hat{\mathcal{G}}$ with utilities $\{\hat{u}^m\}_{m \in \mathcal{M}}$ such that

$$\hat{u}^m(\mathbf{p}) = \begin{cases} 0 & \mathbf{p} \in N_{\mathcal{G}} \\ u^m(\mathbf{p}) - u^m(\mathbf{q}) & \text{if there exists a } \mathbf{q} \in N_{\mathcal{G}} \text{ which is } m\text{-comparable with } \mathbf{p} \\ u^m(\mathbf{p}) & \text{otherwise.} \end{cases} \tag{54}$$

for all $m \in \mathcal{M}$, $\mathbf{p} \in E$. Note that $\hat{u}^m$ is well defined since in $\mathcal{G}$ player m gets the same payoff in all $\mathbf{p} \in N_{\mathcal{G}}$ that are m-comparable. Note that in $\hat{\mathcal{G}}$ all players receive zero payoff at all strategy profiles $\mathbf{p} \in N_{\mathcal{G}}$. To prove the claim, it suffices to show that $\mathcal{G}$ and $\hat{\mathcal{G}}$ have the same potential and harmonic components, or equivalently the game with utilities $\{u^m - \hat{u}^m\}_{m\in\mathcal{M}}$ is nonstrategic, i.e., belongs to $\mathcal{N}$.

In order to prove that the difference is nonstrategic, we first show that the pairwise comparisons of games with utilities $\{u^m\}_{m\in\mathcal{M}}$ and $\{\hat{u}^m\}_{m\in\mathcal{M}}$ are the same. Note that by (54) given m-comparable $\mathbf{p}$ and $\mathbf{q}$, $u^m(\mathbf{p})-u^m(\mathbf{q}) = \hat{u}^m(\mathbf{p})-\hat{u}^m(\mathbf{q})$, if there is no $\mathbf{r} \in N_{\mathcal{G}}$ that is m-comparable with $\mathbf{p}$ or $\mathbf{q}$. If there exists $\mathbf{r} \in N_{\mathcal{G}}$ that is m-comparable with $\mathbf{p}$, then it is also m-comparable with $\mathbf{q}$, hence it follows by (54) that $\hat{u}^m(\mathbf{p})-\hat{u}^m(\mathbf{q}) = u^m(\mathbf{p})-u^m(\mathbf{r})-u^m(\mathbf{q})+u^m(\mathbf{r}) = u^m(\mathbf{p})-u^m(\mathbf{q})$. Note that these equalities hold even if $\mathbf{p}$ or $\mathbf{q}$ is in $N_{\mathcal{G}}$.

Thus, for any m-comparable $\mathbf{p}$ and $\mathbf{q}$ it follows that

$$(u^m(\mathbf{p}) - \hat{u}^m(\mathbf{p})) - (u^m(\mathbf{q}) - \hat{u}^m(\mathbf{q})) = 0,$$

hence the game with utilities $\{u^m - \hat{u}^m\}_{m\in\mathcal{M}}$ is nonstrategic and the claim follows. □

Note that if two games differ only in their nonstrategic components, the pairwise comparisons, and hence the equilibria of these games are identical. Therefore, an immediate implication of the above lemma is that for a given game there exists another game with same potential and harmonic components such that the payoffs at all Nash equilibria are equal to zero. We use this to prove the following Pareto optimality result.

Theorem 5.9. *Let $\mathcal{G}$ be a game with utilities $\{u^m\}$. There exists a game $\bar{\mathcal{G}}$ with utilities $\{\bar{u}^m\}$ such that (i) the potential and harmonic components of $\bar{\mathcal{G}}$ are identical to these of $\mathcal{G}$ and (ii) in $\bar{\mathcal{G}}$ the set of pure NE coincides with the set of Pareto optimal strategy profiles.*

Proof. Games that differ only in nonstrategic components have identical pairwise comparisons, hence the set of Nash equilibria (NE) is the same for such games. Let $N_{\mathcal{G}}$ denote the set of pure NE of $\mathcal{G}$, or equivalently the set of pure NE of a game which differs from $\mathcal{G}$ only by its nonstrategic component.

By Lemma 5.4, it follows that for any game $\mathcal{G}$ there exists a game such that the two games differ only in their nonstrategic components and all players receive zero payoffs at all pure NE (strategy profiles in $N_{\mathcal{G}}$). Therefore, without loss of generality, we let $\mathcal{G}$ be a game in which all players receive zero payoffs at all NE. Given such a game, let $\alpha = 1 + \max_{m,\mathbf{p}} u^m(\mathbf{p})$. Consider the game $\bar{\mathcal{G}}$ with utilities $\{\bar{u}^m\}_{m\in\mathcal{M}}$ such that

$$\bar{u}^m(\mathbf{p}) = \begin{cases} u^m(\mathbf{p}) & \text{if } \mathbf{p} \in N_{\mathcal{G}} \text{ or if there exists a } \mathbf{q} \in N_{\mathcal{G}} \text{ which is } m\text{-comparable with } \mathbf{p} \\ u^m(\mathbf{p}) - \alpha & \text{otherwise.} \end{cases}$$

for all $m \in \mathcal{M}$, $\mathbf{p} \in E$.

Consider m-comparable strategy profiles $\mathbf{p}$ and $\mathbf{q}$. Observe that if there exists a strategy profile $\mathbf{r}$ that is m-comparable with $\mathbf{p}$, it is also m-comparable with $\mathbf{q}$ since by definition of m-comparable strategy profiles $\mathbf{p}^{-m} = \mathbf{r}^{-m} = \mathbf{q}^{-m}$.

Assume that there is a NE that is m-comparable with $\mathbf{p}$ or $\mathbf{q}$, then by definition of $\bar{u}^m$ it follows that $u^m(\mathbf{p}) - u^m(\mathbf{q}) = \bar{u}^m(\mathbf{p}) - \bar{u}^m(\mathbf{q})$. On the contrary if there is no NE that is m-comparable with $\mathbf{p}$ or $\mathbf{q}$ then $\bar{u}^m(\mathbf{p}) - \bar{u}^m(\mathbf{q}) = u^m(\mathbf{p}) - \alpha - u^m(\mathbf{q}) + \alpha = u^m(\mathbf{p}) - u^m(\mathbf{q})$. Hence $\mathcal{G}$ and $\bar{\mathcal{G}}$ have identical pairwise comparisons, and thus the game with utilities $\{u^m - \bar{u}^m\}_{m\in\mathcal{M}}$ is nonstrategic.

We prove the claim, by showing that at all strategy profiles that are not an equilibrium in $\bar{\mathcal{G}}$ (equivalently in $\mathcal{G}$), the players receive nonpositive payoffs and at least one player receives negative payoff and at all NE all players receive zero payoff. This immediately implies that strategy profiles, that are not NE cannot be Pareto optimal, as deviation to a NE increases the payoff of at least one player and the payoff of other players do not decrease by such a deviation. Additionally, it implies that all NE are Pareto optimal, since at all NE all players receive the same payoff, and deviation to a strategy profile that is not a NE strictly decreases the payoff of at least a single player.

By construction it follows that at all NE all players receive zero payoff. Let $\mathbf{p}$ be a strategy profile that is not a NE. If there is some m for which $\mathbf{p}$ is not m-comparable to a NE, then it follows that $\bar{u}^m(\mathbf{p}) = u^m(\mathbf{p}) - \alpha \leq -1$. If on the other hand, $\mathbf{p}$ is m-comparable to a NE, then $\bar{u}^m(\mathbf{p}) \leq 0$, since payoffs are equal to zero at NE. Thus, at any strategy profile, $\mathbf{p}$, that is not a NE players receive nonpositive payoffs, and additionally if for some player m, $\mathbf{p}$ is not m-comparable to a NE, player m receives strictly negative payoff.

To finish the proof we need to show that if $\mathbf{p}$ is m-comparable to a NE for all $m \in \mathcal{M}$, then it still follows that $\bar{u}^m(\mathbf{p}) < 0$ for some $m \in \mathcal{M}$. Assume that this is not true and $\bar{u}^m(\mathbf{p}) = 0$ for all $m \in \mathcal{M}$. Since $\mathbf{p}$ is not a NE, there is at least one player, say m, who can get strictly positive payoff by deviating to a different strategy profile. Therefore this player has strictly positive payoff after its deviation. However, as argued earlier payoffs are nonpositive at strategy profiles that are not NE, and zero at NE. Thus. we reach a contradiction and $\bar{u}^m(\mathbf{p}) < 0$ for some $m \in \mathcal{M}$.

Therefore, it follows that all players have zero payoffs at all NE, and at any other strategy profile all players have nonpositive payoffs and at least one player has strictly negative payoff. □

Note that it is possible to obtain similar results for other efficiency measures using similar arguments to those given in this section. This direction will not be pursued in this paper. The above theorem suggests that the difference in the nonstrategic component of games that are otherwise identical may cause the efficiency properties of these game to be very different. In particular, in one of the games all equilibria may be Pareto optimal when this is not the case for the other game. Therefore, although the nonstrategic component does not change the pairwise comparisons and equilibrium properties in a game it plays a key role in Pareto optimality of equilibria.

5.4 Zero-Sum Games and Identical Interest Games

In this section we present a different decomposition of the space of games, and discuss its relation to our decomposition. To simplify the presentation, we focus on bimatrix games, where each player has h strategies. Before introducing the decomposition, we define zero-sum games and identical interest games.

Definition 5.4 (Zero-sum and Identical Interest Games)**.** *Let $\mathcal{G}$ denote the bimatrix game with payoff matrices (A, B). $\mathcal{G}$ is a* zero-sum game*, if $A + B = 0$, and $\mathcal{G}$ is an* identical interest game*, if $A = B$.*

We denote the set of zero-sum games by $\mathcal{Z}$, and the set of identical interest games by $\mathcal{I}$. Since these sets are defined by equality constraints on the payoff matrices, it follows that they are subspaces.

The idea of decomposing a game to an identical interest game and a zero-sum game was previously mentioned in the literature for two-player games, [1]. The following lemma implies that $\mathcal{Z}$ and $\mathcal{I}$ decomposition of the set of games, has the direct sum property.

Lemma 5.5. *The space of two-player games $\mathcal{G}_{\mathcal{M},E}$ is a direct sum of subspaces of zero-sum and identical interest games, i.e., $\mathcal{G}_{\mathcal{M},E} = \mathcal{Z} \oplus \mathcal{I}$.*

Proof. Consider a bimatrix game with utilities (u^1, u^2). Observe that this game can be decomposed to the games with payoff functions $(\frac{u^1-u^2}{2}, \frac{u^2-u^1}{2})$ and $(\frac{u^1+u^2}{2}, \frac{u^1+u^2}{2})$. Clearly the former game is a zero-sum game, where the latter is an identical interest game. Since the initial game was arbitrary, it follows that any game can be decomposed to a zero-sum game and an identical interest game. The direct sum property follows, since for two-player zero-sum and identical interest games, with utility functions $(u, -u)$ and (v, v) respectively, if $(u+v, u-v) = (0, 0)$, then $u = v = 0$. $\square$

Note that Theorem 5.5 suggests that two-player normalized harmonic games, where players have equal number of strategies are zero-sum.[5] Also, it immediately follows by checking the definitions that identical interest games are potential games. This intuitively suggests that the zero-sum and identical interest game decomposition closely relates to our decomposition. In the following theorem, we establish this relation by characterizing the dimensions of the intersections of the subspaces $\mathcal{Z}$ and $\mathcal{I}$, with the sets of potential and harmonic games. We provide a proof in the Appendix.

Theorem 5.10. *Consider two-player games, in which each player has h strategies. The dimensions of intersections of the subspaces of zero-sum and identical interest games ($\mathcal{Z}$ and $\mathcal{I}$) with the subspaces of potential and harmonic games ($\mathcal{P} \oplus \mathcal{N}$ and $\mathcal{H} \oplus \mathcal{N}$) are as in the following table.*

	$\mathcal{Z}$	$\mathcal{I}$	$\mathcal{Z} \oplus \mathcal{I}$
$\mathcal{P} \oplus \mathcal{N}$	$2h-1$	h^2	h^2+2h-1
$\mathcal{H} \oplus \mathcal{N}$	h^2-2h+2	1	h^2+1
$\mathcal{P} \oplus \mathcal{H} \oplus \mathcal{N}$	h^2	h^2	$2h^2$

Table 5: Dimensions of subspaces of games and their intersections

The above theorem suggests that the dimensions of harmonic games and zero-sum games (and similarly identical interest games and potential games) are close to the dimension of their intersections. Thus, zero-sum games are in general closely related to harmonic games, and identical interest games are related to potential games. On the other hand, it is possible to find instances of zero-sum games that are potential games, and not harmonic games (see Table 6).

	a	b
x	0, 0	1,-1
y	-1, 1	0, 0

(a) Payoffs

	a	b
x	2	1
y	1	0

(b) Potential function

Table 6: A zero-sum potential game

In general, the identical interest component is a potential game, and it can be used to approximate a given game with a potential game. However, as illustrated in Table 7, this approximation need not yield the closest potential game to a given game. In this example, despite the fact that the original game is a potential game, the zero-sum and identical interest game decomposition may lead to a potential game which is much farther than the closest potential game

We believe that the decomposition presented in Section 4 is more natural than the zero-sum identical interest game decomposition, as it clearly separates the strategic ($\mathcal{P} \oplus \mathcal{H}$) and nonstrategic ($\mathcal{N}$) components of games and further identifies components, such as potential and harmonic components, with distinct strategic properties. In addition, it is invariant under trivial manipulations that do not change the strategic interactions, i.e., changes in the nonstrategic component.

[5] In addition, if the definition of zero-sum is generalized to include multiplayer games where payoffs of all players add up to zero, then it can be seen that normalized harmonic games where players have equal number of strategies are still zero-sum games.

	a	b
x	1, 1	1,-1
y	-1, 1	-1, -1

(a) Payoffs in $\mathcal{G}$

	a	b
x	4	2
y	2	0

(b) Potential function of $\mathcal{G}$

	a	b
x	0, 0	1,-1
y	-1, 1	0, 0

(c) Payoffs in $\mathcal{G}_Z$

	a	b
x	1, 1	0,0
y	0, 0	-1, -1

(d) Payoffs in $\mathcal{G}_I$

Table 7: A potential game $\mathcal{G}$ and its zero-sum ($\mathcal{G}_Z$) and identical interest components ($\mathcal{G}_I$).

6 Projections onto Potential and Harmonic Games

In this section, we discuss projections of games onto the subspaces of potential and harmonic games. In Section 4.2, we defined the subspaces $\mathcal{P}, \mathcal{H}, \mathcal{N}$ of potential, harmonic, and nonstrategic components, respectively. We also proved that they provide a direct sum decomposition of the space of all games. In this section, we show that under an appropriately defined inner product in $\mathcal{G}_{\mathcal{M},E}$, the harmonic, potential and nonstrategic subspaces become *orthogonal*. We use our decomposition result together with this inner product to obtain projections of games to these subspaces, i.e., for an arbitrary game, we present closed-form expressions for the "closest" potential and harmonic games with respect to this inner product.

Let $\mathcal{G}, \hat{\mathcal{G}}$ be two games in $\mathcal{G}_{\mathcal{M},E}$. We define the inner product on $\mathcal{G}_{\mathcal{M},E}$ as

$$\langle \mathcal{G}, \hat{\mathcal{G}} \rangle_{\mathcal{M},E} \triangleq \sum_{m \in \mathcal{M}} h_m \langle u^m, \hat{u}^m \rangle, \tag{55}$$

where the inner product in the right hand side is the inner product of C_0 as defined in (7), i.e., it is the inner product of the space of functions defined on E. Note that it can be easily checked that (55) is an inner product, by observing that it is a weighted version of the standard inner product in C_0^M. The given inner product also induces a norm which will help us quantify the distance between games. We define the norm on $\mathcal{G}_{\mathcal{M},E}$ as follows:

$$||\mathcal{G}||^2_{\mathcal{M},E} = \langle \mathcal{G}, \mathcal{G} \rangle_{\mathcal{M},E}. \tag{56}$$

Note that this norm also corresponds to a weighted l_2 norm defined on the space C_0^M.

Next we prove that the potential, harmonic and nonstrategic subspaces are orthogonal under this inner product.

Theorem 6.1. *Under the inner product introduced in* (55)*, we have* $\mathcal{P} \perp \mathcal{H} \perp \mathcal{N}$*, i.e., the potential, harmonic and nonstrategic subspaces are orthogonal.*

Proof. Let $\{u_P^m\}_{m \in \mathcal{M}} = \mathcal{G}_P \in \mathcal{P}$, $\{u_H^m\}_{m \in \mathcal{M}} = \mathcal{G}_H \in \mathcal{H}$ and $\{u_N^m\}_{m \in \mathcal{M}} = \mathcal{G}_N \in \mathcal{N}$ be arbitrary games in $\mathcal{P}$, $\mathcal{H}$ and $\mathcal{N}$ respectively. In order to prove the claim we will first prove $\mathcal{G}_N \perp \mathcal{G}_H$ and $\mathcal{G}_N \perp \mathcal{G}_P$. Secondly we prove $\mathcal{G}_P \perp \mathcal{G}_H$. Since the games are arbitrary the first part will imply that $\mathcal{N} \perp \mathcal{H}$ and $\mathcal{N} \perp \mathcal{P}$ and the second part will imply that $\mathcal{P} \perp \mathcal{H}$ proving the claim.

Note that by definition $u_N^m \in \ker(D_m)$ for all $m \in \mathcal{M}$ and u_P^m, u_N^m are in the orthogonal complement of $\ker(D_m)$ since $\Pi_m u_P^m = u_P^m$, $\Pi_m u_H^m = u_H^m$ and Π_m is the projection operator to the orthogonal complement of $\ker(D_m)$. This implies that $\langle u_P^m, u_N^m \rangle = \langle u_H^m, u_N^m \rangle = 0$ for all $m \in \mathcal{M}$ and hence using the inner product introduced in (55) it follows that $\mathcal{G}_N \perp \mathcal{G}_H$ and $\mathcal{G}_N \perp \mathcal{G}_P$.

Next observe that for all $m \in \mathcal{M}$,

$$\langle u_P^m, u_H^m \rangle = \langle D_m^\dagger D_m u_P^m, u_H^m \rangle \quad = \frac{1}{h_m} \langle D_m^* D_m \phi, u_H^m \rangle = \frac{1}{h_m} \langle \phi, D_m^* D_m u_H^m \rangle,$$

where the first equality follows from $\Pi_m u_P^m = u_P^m$, and the second equality follows from Lemma 4.1 and the fact that $D_m u_P^m = D_m \phi$. The third equality uses the properties of the operators D_m and D_m^*. Therefore,

$$\langle \mathcal{G}_P, \mathcal{G}_H \rangle_{\mathcal{M},E} = \sum_{m\in\mathcal{M}} \langle \phi, D_m^* D_m u_H^m \rangle \;\; = \langle \phi, \sum_{m\in\mathcal{M}} D_m^* D_m u_H^m \rangle = \langle \phi, \delta_0^* \sum_{m\in\mathcal{M}} D_m u_H^m \rangle = 0.$$

Since $\delta_0^* \sum_{m\in\mathcal{M}} D_m u_H^m = 0$ by the definition of $\mathcal{H}$. Here the last equality follows using $\delta_0^* = \sum_m D_m^*$ and orthogonality of the image spaces of D_m for $m \in \mathcal{M}$. Therefore, $\mathcal{G}_H \perp \mathcal{G}_P$ as claimed and the result follows. □

The next theorem provides closed form expressions for the closest potential and harmonic games with respect to the norm in (56).

Theorem 6.2. *Let $\mathcal{G} \in \mathcal{G}_{\mathcal{M},E}$ be a game with utilities $\{u^m\}_{m\in\mathcal{M}}$, and let $\phi = \delta_0^\dagger D u$. With respect to the norm in* (56)*,*

1. *The closest potential game to $\mathcal{G}$ has utilities $\Pi_m \phi + (I - \Pi_m) u^m$ for all $m \in \mathcal{M}$,*
2. *The closest harmonic game to $\mathcal{G}$ has utilities $u^m - \Pi_m \phi$ for all $m \in \mathcal{M}$.*

Proof. By Theorem 6.1, the harmonic component of $\mathcal{G}$ is orthogonal to the space of potential games $\mathcal{P} \oplus \mathcal{N}$. Thus, the closest potential game to $\mathcal{G}$ has utilities $u^m - u_H^m$, where $\{u_H^m\}_{m\in\mathcal{M}}$ is the harmonic component of $\mathcal{G}$. Similarly, the potential component of $\mathcal{G}$ is orthogonal to the space of harmonic games $\mathcal{H} \oplus \mathcal{N}$ and thus the closest harmonic game to $\mathcal{G}$ has utilities $u^m - u_P^m$, where $\{u_P^m\}_{m\in\mathcal{M}}$ is the potential component of $\mathcal{G}$. Using the closed form expressions for u_P^m and u_H^m from Theorem 4.1, the claim follows. □

Note that the utilities in the closest potential game consist of two parts: the term $\Pi_m \phi$ expresses the preferences that are captured by the potential function ϕ, and $(I - \Pi_m) u^m$ corresponds to the nonstrategic component of the original game. Similarly, the closest harmonic game differs from the original game by its potential component, and hence has the same nonstrategic and harmonic components with the original game. This implies that the projection decomposes the flows generated by a game to its consistent and inconsistent components and is closely related to the decomposition of flows to the orthogonal subspaces of the space of flows provided in the Helmholtz decomposition.

Analyzing the projection of a game to the space of potential games may provide useful insights for the original game; see Section 7 for a description of ongoing and future work on this direction. We conclude this section by relating the approximate equilibria of a game to the equilibria of the closest potential game.

Theorem 6.3. *Let $\mathcal{G}$ be a game, and $\hat{\mathcal{G}}$ be its closest potential game. Assume that h_m denotes the number of strategies of player m, and define $\alpha \triangleq ||\mathcal{G} - \hat{\mathcal{G}}||_{\mathcal{M},E}$. Then, every ϵ_1-equilibrium of $\hat{\mathcal{G}}$ is an ϵ-equilibrium of $\mathcal{G}$ for some $\epsilon \leq \max_m \frac{2\alpha}{\sqrt{h_m}} + \epsilon_1$ (and viceversa).*

Proof. By the definition of the norm, it follows that

$$|u^k(\mathbf{p}) - \hat{u}^k(\mathbf{p})| \leq \frac{1}{\sqrt{h_k}} ||\mathcal{G} - \hat{\mathcal{G}}||_{\mathcal{M},E} \leq \max_m \frac{\alpha}{\sqrt{h_m}},$$

for all $k \in \mathcal{M}$, $\mathbf{p} \in E$. Using Lemma 2.1, the result follows. □

This result implies that the study and characterization of the structure of approximate equilibria in an arbitrary game can be facilitated by making use of the connection between its ϵ-equilibrium set and the equilibria of its closest potential game.

7 Conclusions

We have introduced a novel and natural direct sum decomposition of the space of games into potential, harmonic and nonstrategic subspaces. We studied the equilibrium properties of the subclasses of games induced by this decomposition, and showed that the potential and harmonic components of games have quite distinct and appealing equilibrium properties. In particular, there is a sharp contrast between potential games, that always have pure Nash equilibria, and harmonic games, that generically never do. Moreover, we have shown that while the nonstrategic component does not effect the equilibrium set of games, it can drastically affect their efficiency properties. Using the decomposition framework, we obtained closed-form expressions for the projections of games to their corresponding components, enabling the approximation of arbitrary games in terms of potential and harmonic games. This provides a systematic method for characterizing the set of ϵ-equilibria by relating it to the equilibria of the closest potential game.

The framework provided in this paper opens up a number of interesting research directions, several of which we are currently investigating. Among them, we mention the following:

Decomposition and dynamics One immediate and promising direction is to use the projection techniques to analyze natural player dynamics through the convergence properties of their potential component. It is well-known that in potential games many natural dynamics, such as best-response and fictitious play, converge to an equilibrium [47, 27]. In our companion paper [3], we show that such dynamics converge to a neighborhood of equilibria in near-potential games, where the size of the neighborhood depends on the distance of the original game to its closest potential game.

Dynamics in harmonic games While the behavior of player dynamics in potential games is reasonably well-understood, there seems to be a number of interesting research questions regarding their harmonic counterpart. In [3], we made some partial progress in this direction, by showing that in harmonic games, the uniformly mixed strategy profile is the unique equilibrium of the continuous time fictitious-play dynamics and this equilibrium point is locally stable. Moreover, in two-player games where each player has equal number of strategies, this equilibrium is globally stable. Global stability of the equilibrium under more general settings and convergence of different dynamics in harmonic games are open future questions.

Game approximation The idea of analyzing an arbitrary game through a "nearby" game with tractable equilibrium properties seems to be a useful approach. In [4], we have developed this methodology ("near-potential" games), and applied it in the context of pricing in a networking application. We believe that these techniques can be extended to other special classes of games. An extension to generalizations of potential games (weighted and ordinal potential games) was considered in [5]. Another interesting direction is to study the proximity of an arbitrary game to supermodular games [43] and analyze how properties of supermodular games are inherited in "near-supermodular" games.

Alternative projections In this work, the projections onto the spaces of potential and harmonic games are obtained using a weighted l_2 norm. This norm leads to closed-form expressions for the components, but some problems may require or benefit from projections using different metrics. For instance, finding the closest potential game, by perturbing each of the pairwise comparisons in a minimal way requires a projection using a suitably defined ∞-norm, and this projection leads to better error bounds when analyzing approximate equilibria and dynamics. Projections under different norms and their properties are left for future research.

Additional restrictions Another interesting extension is to study projections to subsets of potential games with additional properties. For example, in the current projection framework, if we require the potential function to be concave (see [44] for a definition of discrete concavity), it may be possible to project a given game to a set of potential games with a unique Nash equilibrium.

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References

[1] T. Başar and Y.C. Ho. Informational properties of the Nash solutions of two stochastic nonzero-sum games. *Journal of Economic Theory*, 7(4):370–387, 1974.

[2] D. Bertsimas and J. Tsitsiklis. *Introduction to Linear Optimization.* Athena Scientific, 1997.

[3] O. Candogan, I. Menache, A. Ozdaglar, and P. A. Parrilo. Dynamics in games and near potential games. Technical report, LIDS, MIT, November 2009.

[4] O. Candogan, I. Menache, A. Ozdaglar, and P. A. Parrilo. Near-optimal power control in wireless networks: a potential game approach. *INFOCOM 2010 - The 29th IEEE Conference on Computer Communications.*, 2010.

[5] O. Candogan, A. Ozdaglar, and P. A. Parrilo. A projection framework for near-potential games. Technical report, LIDS, MIT, April 2010.

[6] G. Christodoulou, V.S. Mirrokni, and A. Sidiropoulos. Convergence and approximation in potential games. *Lecture Notes in Computer Science*, 3884:349, 2006.

[7] F.R.K. Chung. *Spectral graph theory.* American Mathematical Society, 1997.

[8] R. Cressman and W.G. Morrison. On the evolutionary dynamics of crime. *The Canadian Journal of Economics/Revue canadienne d'Economique*, 31(5):1101–1117, 1998.

[9] D. Friedman. Evolutionary games in economics. *Econometrica*, 59(3):637, 1991.

[10] D. Fudenberg and D.K. Levine. *The theory of learning in games.* MIT Press, 1998.

[11] D. Fudenberg and J. Tirole. *Game Theory.* MIT Press, 1991.

[12] F. Germano. On some geometry and equivalence classes of normal form games. *International Journal of Game Theory*, 34(4):561–581, 2006.

[13] I. Gilboa and D. Schmeidler. Canonical representation of set functions. *Mathematics of Operations Research*, 20(1):197–212, 1995.

[14] M. Goemans, V. Mirrokni, and A. Vetta. Sink equilibria and convergence. In *FOCS '05: Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science*, pages 142–154, Washington, DC, USA, 2005. IEEE Computer Society.

[15] G.H. Golub and C.F. Van Loan. *Matrix computations.* Johns Hopkins University Press, 1996.

[16] P.J. Hammond. Utility invariance in non-cooperative games. *Advances in public economics: utility, choice and welfare*, pages 31–50, 2005.

[17] J. Hillas and E. Kohlberg. Foundations of strategic equilibrium. *Handbook of game theory*, 3:1597–1663, 2002.

[18] J. Hofbauer and E. Hopkins. Learning in perturbed asymmetric games. *Games and Economic Behavior*, 52(1):133–152, 2005.

[19] J. Hofbauer and W.H. Sandholm. On the global convergence of stochastic fictitious play. *Econometrica*, pages 2265–2294, 2002.

[20] J. Hofbauer and K.H. Schlag. Sophisticated imitation in cyclic games. *Journal of Evolutionary Economics*, 10(5):523–543, 2000.

[21] X. Jiang, L.H. Lim, Y. Yao, and Y. Ye. Statistical ranking and combinatorial Hodge theory. arXiv:0811.1067, 2008.

[22] J.S. Jordan. Three problems in learning mixed-strategy Nash equilibria. *Games and Economic Behavior*, 5(3):368–386, 1993.

[23] R. Kannan and T. Theobald. Games of fixed rank: a hierarchy of bimatrix games. *Economic Theory*, 42(1):157–173, 2010.

[24] N.L Kleinberg and J. H. Weiss. Equivalent n-person games and the null space of the Shapley value. *Mathematics of Operations Research*, 10(2):233–243, 1985.

[25] N.L. Kleinberg and J.H. Weiss. Algebraic structure of games. *Mathematical Social Sciences*, 9(1):35–44, 1985.

[26] N.L. Kleinberg and J.H. Weiss. The orthogonal decomposition of games and an averaging formula for the Shapley value. *Mathematics of Operations Research*, pages 117–124, 1986.

[27] J.R. Marden, G. Arslan, and J.S. Shamma. Joint strategy fictitious play with inertia for potential games. *IEEE Transactions on Automatic Control*, 54(2):208–220, 2009.

[28] J.R. Marden and J.S. Shamma. Revisiting log-linear learning: Asynchrony, completeness and a payoff-based implementation. *Games and Economic Behavior*, submitted, 2008.

[29] M. Marinacci. Decomposition and representation of coalitional games. *Mathematics of Operations Research*, pages 1000–1015, 1996.

[30] J.F. Mertens. Ordinality in non cooperative games. *International Journal of Game Theory*, 32(3):387–430, 2004.

[31] D. Monderer and L.S. Shapley. Fictitious play property for games with identical interests. *Journal of Economic Theory*, 68(1):258–265, 1996.

[32] D. Monderer and L.S. Shapley. Potential games. *Games and Economic Behavior*, 14(1):124–143, 1996.

[33] S. Morris and T. Ui. Best response equivalence. *Games and Economic Behavior*, 49(2):260–287, 2004.

[34] H. Moulin and J.P. Vial. Strategically zero-sum games: The class of games whose completely mixed equilibria cannot be improved upon. *International Journal of Game Theory*, 7(3):201–221, 1978.

[35] A. Neyman. Correlated equilibrium and potential games. *International Journal of Game Theory*, 26(2):223–227, 1997.

[36] K. Polthier and E. Preuß. Identifying vector field singularities using a discrete Hodge decomposition. *Visualization and Mathematics III*, 2002.

[37] R. Rosenthal. Correlated equilibria in some classes of two-person games. *International Journal of Game Theory*, 3(3):119–128, 1974.

[38] W.H. Sandholm. Decompositions and potentials for normal form games. *Unpublished manuscript, University of Wisconsin*, 2008.

[39] J.S. Shamma and G. Arslan. Unified convergence proofs of continuous-time fictitious play. *IEEE Transactions on Automatic Control*, 49(7):1137–1141, 2004.

[40] L. Shapley. A value for n-person games. *In Contributions to the Theory of Games, II. ed. H. W. Kuhn and A. W. Tucker*, pages 307–317, 1953.

[41] J.M. Smith and J. Hofbauer. The "battle of the sexes": a genetic model with limit cycle behavior. *Theoretical population biology*, 32(1):1–14, 1987.

[42] Y. Tong, S. Lombeyda, A.N. Hirani, and M. Desbrun. Discrete multiscale vector field decomposition. *ACM Transactions on Graphics*, 22(3):445–452, 2003.

[43] D.M. Topkis. *Supermodularity and Complementarity.* Princeton University Press, 1998.

[44] T. Ui. Discrete concavity for potential games. *International Game Theory Review*, 10(01):137–143, 2008.

[45] J. Von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior.* Princeton University Press, 1947.

[46] M. Voorneveld. Best-response potential games. *Economics Letters*, 66(3):289–295, 2000.

[47] H.P. Young. *Strategic Learning and Its Limits.* Oxford University Press, 2004.

A Additional Proofs

In this section we provide proofs to some of the results from Sections 4 and 5.

Proof of Lemmas 4.1 and 4.2. The proof relies on the fact that $D_m^* D_m = \Delta_{0,m}$ is a Laplacian operator defined on the graph of m-comparable strategy profiles. We show that the kernels of D_m and $\Delta_{0,m}$ coincide, and using the spectral properties of the Laplacian and projection matrices we obtain the desired result.

For a fixed m, it can be seen that strategy profile $\mathbf{p} = (\mathbf{p}^m, \mathbf{p}^{-m})$ is comparable to strategy profiles $(\mathbf{q}^m, \mathbf{p}^{-m})$ for all $\mathbf{q}^m \in E^m$, $\mathbf{q}^m \neq \mathbf{p}^m$ but to none of the strategy profiles $(\mathbf{q}^m, \mathbf{q}^{-m})$ for $\mathbf{q}^{-m} \neq \mathbf{p}^{-m}$. This implies that the graph over which $\Delta_{0,m}$ is defined has $|E^{-m}| = \prod_{k \neq m} h_k$ components (each $\mathbf{p}^{-m} \in E^{-m}$ creates a different component), each of which has $|E^m| = h_m$ elements. Note that all strategy profiles in a component are m-comparable, thus the underlying graph consists of $|E^{-m}|$ components, each of which is a complete graph with $|E^m|$ nodes.

The Laplacian of an unweighted complete graph with n nodes has eigenvalues 0 and n, where the multiplicity of nonzero eigenvalues is $n-1$ [7]. Each component of $\Delta_{0,m}$ leads to eigenvalues 0 and h_m with multiplicities 1 and $h_m - 1$ respectively. Therefore, $\Delta_{0,m}$ has eigenvalues 0 and h_m where the multiplicity of nonzero eigenvalues is $(h_m - 1)\prod_{k \neq m} h_k = \prod_k h_k - \prod_{k \neq m} h_k$. This suggests that the dimension of the kernel of $\Delta_{0,m}$ is $\prod_{k \neq m} h_k$.

Observe that the kernel of $\Delta_{0,m} = D_m^* D_m$ contains the kernel of D_m. For every $\mathbf{q}^{-m} \in E^{-m}$ define $\nu_{\mathbf{q}^{-m}} \in C_0$ such that

$$\nu_{\mathbf{q}^{-m}}(\mathbf{p}) = \begin{cases} 1 & \text{if } \mathbf{p}^{-m} = \mathbf{q}^{-m} \\ 0 & \text{otherwise} \end{cases} \tag{57}$$

It is easy to see that $\nu_{\mathbf{p}^{-m}} \perp \nu_{\mathbf{q}^{-m}}$ for $\mathbf{p}^{-m} \neq \mathbf{q}^{-m}$ and $D_m \nu_{\mathbf{p}^{-m}} = 0$ for all $\mathbf{p}^{-m} \in E^{-m}$. Thus, for all $\mathbf{q}^{-m}$, $\nu_{\mathbf{q}^{-m}}$ belongs to the kernel of D_m and by mutual orthogonality of these functions, the kernel of D_m has dimension at least $|E^{-m}| = \prod_{k \neq m} h_k$. As the dimension of the kernel of $\Delta_{0,m}$ is $\prod_{k \neq m} h_k$ and it contains kernel of D_m, this implies that the kernels of D_m and $\Delta_{0,m}$ coincide.

Thus $\Delta_{0,m}$ maps any $\nu \in C_0$ in the kernel of D_m to zero and scales the ν in the orthogonal complement of the kernel by h_m. On the other hand $D_m^\dagger D_m$ is a projection operator and it has eigenvalue 0 for all functions in the kernel of D_m and 1 for the functions in the orthogonal complement of kernel of D_m. This implies that

$$\Delta_{0,m} = h_m D_m^\dagger D_m, \tag{58}$$

and the kernels of Π_m, D_m and $\Delta_{0,m}$ coincide as the claim suggests. $\square$

Proof of Lemma 4.3. For a game, the graph of comparable strategy profiles is connected as can be seen from the definition of the comparable strategy profiles. It is known that for a connected graph, the Laplacian operator has multiplicity 1 for eigenvalue 0 [7]. By (15) it follows that the function $f \in C_0$ satisfying $f(\mathbf{p}) = 1$ for all $\mathbf{p} \in E$, is an eigenfunction of Δ_0 with eigenvalue 0, implying the result. $\square$

Proof of Lemma 4.4. For the proof of this lemma, we use the following property of the pseudoinverse

$$A^\dagger = (A^* A)^\dagger A^*, \tag{59}$$

and the orthogonality properties of the D_m operators: $D_m^* D_k = 0$ and $D_m^\dagger D_k = 0$ if $m \neq k$.

(i) Using (59), with $A = D_m$ implies that $D_m^\dagger = (D_m^* D_m)^\dagger D_m^*$. Since for any linear operator L, $(L^\dagger)^* = (L^*)^\dagger$, it follows that $D_m^\dagger = (D_m (D_m^* D_m)^\dagger)^* = (D_m(\Delta_{0,m})^\dagger)^*$. Hence, using Lemma 4.1 we obtain $D_m^\dagger = h_m (D_m(\Pi_m)^\dagger)^*$. Since Π_m is a projection operator to the orthogonal complement of the kernel of D_m, we have $\Pi_m^\dagger = \Pi_m$, and $D_m \Pi_m = D_m$. Hence, it follows that $D_m^\dagger = h_m (D_m \Pi_m)^* = h_m D_m^*$ as claimed.

(ii) The identity in (59), implies that

$$(\sum_i D_i)^\dagger = \left((\sum_i D_i)^* (\sum_i D_i) \right)^\dagger (\sum_i D_i)^*.$$

By the orthogonality of the image spaces of D_i, it follows that $(\sum_i D_i)^*(\sum_i D_i) = \sum_i D_i^* D_i$, and hence

$$(\sum_i D_i)^\dagger = \left(\sum_i D_i^* D_i \right)^\dagger (\sum_i D_i)^*.$$

Right-multiplying the above equation by D_j and using the orthogonality of the image spaces of D_is it follows that

$$(\sum_i D_i)^\dagger D_j = \left(\sum_i D_i^* D_i \right)^\dagger (\sum_i D_i)^* D_j = \left(\sum_i D_i^* D_i \right)^\dagger D_j^* D_j.$$

(iii) From the definition of pseudoinverse, it is sufficient to show the following 4 properties to prove the claim: a) $(DD^\dagger)^* = DD^\dagger$, b) $(D^\dagger D)^* = D^\dagger D$, c) $DD^\dagger D = D$, d) $D^\dagger D D^\dagger = D^\dagger$.

Using the identity $D_m^\dagger D_k = 0$ for $k \neq m$, it follows that $DD^\dagger = \sum_m D_m D_m^\dagger$, and $D^\dagger D = diag\left(D_1^\dagger D_1, \dots D_M^\dagger D_M\right)$. The pseudoinverse of D_m satisfies the properties $D_m^\dagger D_m = (D_m^\dagger D_m)^*$ and $D_m D_m^\dagger = (D_m D_m^\dagger)^*$, and the requirements a) and b) follow immediately using these properties. The identity $D_m^\dagger D_k = 0$ also implies that $DD^\dagger D = [D_1 D_1^\dagger D_1, \dots, D_M D_M^\dagger D_M]$, and $D^\dagger D D^\dagger = [D_1^\dagger D_1 D_1^\dagger; \dots; D_M^\dagger D_M D_M^\dagger]$. Since the pseudoinverse of D_m also satisfies $D_m^\dagger D_m D_m^\dagger = D_m^\dagger$, and $D_m D_m^\dagger D_m = D_m$, the requirements c) and d) are satisfied and the claim follows.

(iv) Since $\Pi = diag\,(\Pi_1, \dots \Pi_M)$, and $\Pi_m = D_m^\dagger D_m$, it follows that $D^\dagger D = diag\left(D_1^\dagger D_1, \dots D_M^\dagger D_M\right) = diag\,(\Pi_1, \dots \Pi_M) = \Pi$.

(v) Using the identities $D_m^\dagger D_k = 0$ for $k \neq m$, $\delta_0 = \sum_m D_m$, it follows that

$$DD^\dagger \delta_0 = DD^\dagger \sum_{m \in \mathcal{M}} D_m = \sum_{m \in \mathcal{M}} D_m D_m^\dagger D_m = \sum_{m \in \mathcal{M}} D_m = \delta_0.$$

□

Proof of Lemma 5.2. Let $X = Du$ denote the pairwise comparison function of the harmonic game. By definition, $(\delta_0^* X)(\mathbf{p}) = 0$ for all $\mathbf{p} \in E$. Thus, for all $\mathbf{r}^m \in E^m$, it follows that

$$0 = \sum_{\mathbf{p}^{-m} \in E^{-m}} (\delta_0^* X)(\mathbf{r}^m, \mathbf{p}^{-m}) = \sum_{\mathbf{p} \in S} (\delta_0^* X)(\mathbf{p}) \tag{60}$$

where $S = \{(\mathbf{r}^m, \mathbf{p}^{-m}) | \mathbf{p}^{-m} \in E^{-m}\}$. To complete the proof we require the following identity related to the pairwise comparison functions.

Lemma A.1. *For all $\hat{X} \in C_1$ and set of strategy profiles $\hat{S} \subset E$, $\sum_{\mathbf{p}\in\hat{S}}(\delta_0^*\hat{X})(\mathbf{p}) = -\sum_{\mathbf{p}\in\hat{S}}\sum_{\mathbf{q}\in\hat{S}^c}\hat{X}(\mathbf{p},\mathbf{q})$.*

Proof. It follows from the definition of δ_0^* that

$$\begin{aligned}\sum_{\mathbf{p}\in\hat{S}}(\delta_0^*\hat{X})(\mathbf{p}) &= -\sum_{\mathbf{p}\in\hat{S}}\sum_{\mathbf{q}\in E}\hat{X}(\mathbf{p},\mathbf{q})\\ &= -\sum_{\mathbf{p}\in\hat{S}}\sum_{\mathbf{q}\in\hat{S}^c}\hat{X}(\mathbf{p},\mathbf{q}) - \sum_{\mathbf{p}\in\hat{S}}\sum_{\mathbf{q}\in\hat{S}}\hat{X}(\mathbf{p},\mathbf{q})\\ &= -\sum_{\mathbf{p}\in\hat{S}}\sum_{\mathbf{q}\in\hat{S}^c}\hat{X}(\mathbf{p},\mathbf{q}).\end{aligned} \tag{61}$$

since $\hat{X}(\mathbf{p},\mathbf{q}) + \hat{X}(\mathbf{q},\mathbf{p}) = 0$ for any $\mathbf{p},\mathbf{q}$ and thus $\sum_{\mathbf{p}\in\hat{S}}\sum_{\mathbf{q}\in\hat{S}}\hat{X}(\mathbf{p},\mathbf{q}) = 0$. □

Using this lemma in (60), we obtain

$$\begin{aligned}0 &= -\sum_{\mathbf{p}\in S}\sum_{\hat{\mathbf{p}}\in S^c}X(\mathbf{p},\hat{\mathbf{p}})\\ &= \sum_{\mathbf{p}^{-m}\in E^{-m}}\sum_{\mathbf{p}^m\in E^m}u^m(\mathbf{r}^m,\mathbf{p}^{-m}) - u^m(\mathbf{p}^m,\mathbf{p}^{-m}).\end{aligned} \tag{62}$$

Since $\mathbf{r}^m$ is arbitrary, it follows that

$$\sum_{\mathbf{p}^{-m}\in E^{-m}}u^m(\mathbf{q}^m,\mathbf{p}^{-m}) - u^m(\mathbf{r}^m,\mathbf{p}^{-m}) = 0. \tag{63}$$

for all $\mathbf{q}^m,\mathbf{r}^m \in E^m$. □

Proof of Theorem 5.10. Since $\mathcal{Z}\oplus\mathcal{I} = \mathcal{G}_{\mathcal{M},E}$, the last column immediately follows from Proposition 5.1 and Theorem 4.1. Below, we present the dimension results for each row of the table, and the corresponding entries in the first two columns.

Throughout the proof we denote by e the h dimensional vector of ones. Since $\mathcal{N} = \ker D$, the two-player games in $\mathcal{N}$ take the form (ea^T, be^T) for some $a,b\in\mathbb{R}^h$. We shall make use of this fact in the proof.

$\mathcal{P}\oplus\mathcal{H}\oplus\mathcal{N}$: Since $\mathcal{P}\oplus\mathcal{H}\oplus\mathcal{N} = \mathcal{G}_{\mathcal{M},E}$, it follows that $\dim(\mathcal{P}\oplus\mathcal{H}\oplus\mathcal{N})\cap\mathcal{Z} = \dim\mathcal{Z}$ and $\dim(\mathcal{P}\oplus\mathcal{H}\oplus\mathcal{N})\cap\mathcal{I} = \dim\mathcal{I}$. Zero-sum games are games with payoff matrices $(A,-A)$ for some $A\in\mathbb{R}^{h\times h}$. Thus, the dimension of the zero-sum games is equivalent to the dimension of possible A matrices that define zero-sum games and hence $\dim\mathcal{Z} = h^2$. Similarly, identical interest games are games with payoff matrices (A,A) for some $A\in\mathbb{R}^{h\times h}$, and hence $\dim\mathcal{I} = h^2$.

$\mathcal{P}\oplus\mathcal{N}$: By Theorem 5.1, it follows that $\mathcal{P}\oplus\mathcal{N}$ is equivalent to the set of potential games. Observe that all identical interest games are potential games, where the utility functions of players are equal to the potential function of the game. Thus, it follows that $\dim(\mathcal{P}\oplus\mathcal{N})\cap\mathcal{I} = \dim\mathcal{I} = h^2$.

Let $\mathcal{G}$ denote a zero-sum game in $\mathcal{P}\oplus\mathcal{N}$, with payoff matrices $(A,-A)$, and denote the matrix corresponding to a potential function of $\mathcal{G}$ by ϕ. Thus, both the game with payoffs $(A,-A)$ and (ϕ,ϕ) belong to $\mathcal{N}\oplus\mathcal{P}$, and $(A,-A)$ is different from (ϕ,ϕ) by its nonstrategic component. Hence, for some $a,b\in\mathbb{R}^h$, $A = \phi + ea^T$, $-A = \phi + be^T$, for some $a,b\in\mathbb{R}^h$ and

$$A - A = \phi + ea^T + \phi + be^T = 2\phi + ea^T + be^T = 0,$$

thus $-2\phi_{ij} = a_j + b_i$ and $A_{ij} = \phi_{ij} + a_j = \frac{a_j - b_i}{2}$ for all $i,j\in\{1\dots n\}$. Hence, $a,b\in\mathbb{R}^h$ characterize the possible payoff matrices A, and it can be seen that the set of these matrices has dimension

$2h-1$. Since these matrices uniquely characterize zero-sum games that are also potential games, it follows that the dimension of $(\mathcal{P}\oplus\mathcal{N})\cap\mathcal{Z}$ is equal to $2h-1$.

$\mathcal{H}\oplus\mathcal{N}$: The games in this set do not have potential components. If a game in $\mathcal{H}\oplus\mathcal{N}$ is an identical interest game, then it also belongs to $\mathcal{P}\oplus\mathcal{N}$. Due to the direct sum property of $\mathcal{P}\oplus\mathcal{H}\oplus\mathcal{N}$, it follows that this game can only have nonstrategic component. Therefore, $\dim(\mathcal{H}\oplus\mathcal{N})\cap\mathcal{I} = \dim\mathcal{N}\cap\mathcal{I}$. Let $\mathcal{G}$ denote a game in $\mathcal{N}\cap\mathcal{I}$. Since $\mathcal{G}$ has only nonstrategic information it follows that its payoffs are given by (ea^T, be^T), for some vectors a and b. Then, being an identical interest game implies that $ea^T = be^T$, which requires that all entries of payoff matrices are identical, thus $\dim\mathcal{N}\cap\mathcal{I} = 1$.

Consider a zero-sum game in $\mathcal{H}\oplus\mathcal{N}$, with payoff matrices $(A,-A)$. Since, both players have equal number of strategies, the harmonic component of this game is also zero-sum and the payoff matrices in the harmonic component can be denoted by $(A_H,-A_H)$ for some $A_H\in\mathbb{R}^{h\times h}$. Because the original game is in $\mathcal{H}\oplus\mathcal{N}$, the payoff matrices satisfy $A = A_H + ea^T$, $-A = -A_H + be^T$, where (ea^T, be^T) corresponds to the nonstrategic component of the game. It follows that $ea^T+be^T = 0$, and hence ea^T and $-be^T$ are matrices, which have all of their entries identical. Thus, the nonstrategic component of the games in $(\mathcal{H}\oplus\mathcal{N})\cap\mathcal{Z}$, forms a 1 dimensional subspace. Since the harmonic component is arbitrary, it follows that $\dim(\mathcal{H}\oplus\mathcal{N})\cap\mathcal{Z} = \dim\mathcal{H}+1 = (h-1)^2+1 = h^2-2h+2$. $\square$